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THEORETICAL INVESTIGATION OF SUBSONIC OSCILLATORY BLADE-ROW AERODYNAMICS

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SUMMARY

A method is presented for calculating the aerodynamic lifts and moments experienced by a cascade or two-dimensional approximation to a compressor or turbine blade row in subsonic flow under harmonic oscillation. Arbitrary stagger and interblade phase-lag angles are permitted. The most significant features of the method stem from the utilization of Fourier transforms of blade pressure-jump functions. This permits expression in closed form of the kernel function appearing in the integral equation relating upwash to the pressure-jump transform. So-called resonance phenomena, first discovered in connection with the subsonic wall interference problem, are shown to occur in the case of cascade oscillation even in the presence of stagger and arbitrary interblade phase-lag angles. The resonance points are shown to be related to the poles of the kernel function for the Fourier transform formulation of the problem. Numerical techniques are developed for the direct solution of the transform rather than of the pressure jump proper, and it is shown that lift and moment may be easily expressed in terms of the pressure-jump transform without requiring any inversion. The method is applied to computation of a zero-stagger cascade in antiphase motion (corresponding to the tunnel-wall interference problem) in order to permit comparison of results with those obtained by solution of the more conventional integral equation based upon the Hankel function series type of kernel. Excellent agreement is obtained in all cases.

INTRODUCTION

The aerodynamic forces and moments experienced by compressor blades undergoing oscillatory motion or in the presence of certain classes of oscillatory inflow are of interest to the aircraft-jet-engine industry for several reasons. First, while the occurrence of classical flutter in compressor blades has not been a matter of concern up to this time, it is conceivable that current design trends may lead to configurations and flow conditions in which the avoidance of classical flutter must become a consideration. Second, the value of aerodynamic damping is of interest quite apart from the flutter problem proper, and the existence in compressible subsonic flow of certain so-called resonance phenomena can have a major influence upon this source of damping. Third, the aerodynamic behavior of blade rows under the influence of certain types of inflow disturbance or nonuniformity may be calculated by the same methods utilized in the computation of aerodynamic forces and moments experienced by the blades undergoing oscillations. Thus, a technique for computing subsonic, oscillatory, blade-row air forces and moments would have several areas of application in the compressor field. Moreover, the problem of tunnel-wall interference effects in subsonic wind-tunnel tests for oscillatory aerodynamic pressures constitutes a special case of the blade-row problem as treated herein, namely, the case in which stagger is zero and adjacent blades are 180° out of phase (this being the proper image system equivalent to the wall influence). This problem has been previously treated by the National Advisory Committee for Aeronautics (refs. 1 and 2) by utilizing a form of aerodynamic integral equation which constitutes a natural extension of the equation for the single wing oscillating in subsonic flow. It is believed that the computation method developed herein for the more general case including stagger and arbitrary interblade phase-lag angle may prove more convenient for application to the wall interference problem because of its relatively simple form of kernel function. It is even probable that the wall interference problem for the tunnel with slotted walls, whose image equivalent constitutes the special case of zero stagger and in-phase motion of all blades, may prove most easily calculable from the scheme derived herein.

The present work constitutes a natural extension of previous investigations (refs. 3 and 4) concerning the general problem of oscillatory blade-row behavior under flutter or vibration. These investigations established the blade-system mode shapes which characterize a blade row at flutter (or even in vibration distinct from flutter) and developed the computation techniques for oscillatory air forces in the incompressible flow regime. This report presents an entirely different approach to the problem of calculating aerodynamic forces and moments experienced by cascades or of approximating compressor blade rows in the subsonic two-dimensional regime. Considerations are limited to low incidence, and effects of blade thickness and camber are neglected. The mutual oscillatory aerodynamic interference of all blades is accounted for under the class of system mode shapes the existence of which is proven in reference 3.

By use of the Fourier transform the kernel function of the aerodynamic integral equation is effectively reduced from an extremely cumbersome form involving semi-infinite-range integrals of doubly infinite series of Hankel functions to a simple, single, closed-form expression. Moreover, the nature of the kernel is such that close control over accuracy is facilitated. Concurrently with the study of the formulation and numerical solution of the aerodynamic integral equation considerable attention has been given to the occurrence of the so-called resonance phenomenon. This resonance phenomenon constitutes an infinite set of values of a parameter depending upon flow and configuration characteristics at which the aerodynamic kernel function becomes everywhere infinite. The phenomenon was described and treated in detail by Woolston and Runyan (ref. 1) for the wind-tunnel-wall interference problem and, consequently, could be interpreted as a situation to be encountered by the cascade or blade row at zero stagger and under antiphase blade motion. The present report shows that the resonance condition may occur in staggered cascades with arbitrary interblade phase lag, the resonance parameter values depending upon Mach number, oscillatory frequency, interblade phase angle, gap, stagger, and acoustic velocity.

The resonance phenomenon could be a matter of serious concern to compressor designers in view of the fact that it represents a set of conditions at which all self-induced aerodynamic forces, including the beneficial aerodynamic damping usually present below flutter speed, disappear. The blades act effectively as if they were in a vacuum. Thus, not only will certain critical blade oscillation frequencies be undamped, but disturbances caused by wakes or upstream disturbances giving rise to the proper combination of frequency and interblade phase lag may fall at the resonance condition with consequent disappearance of aerodynamic damping. This resonance-induced damping loss in an actual compressor configuration will, of course, be mitigated by the radial variation of cascade and flow parameters. These variations preclude the possibility of a major radial extent of the blade row becoming simultaneously resonant.

Following the development of the aerodynamic integral equation in the Fourier transform domain, the discussion of resonance, and the formulation of calculation techniques, the results of a specific set of computations are presented in order to provide confidence in the feasibility and accuracy of the method. The conditions for which the calculations are performed correspond to the conditions at which Woolston and Runyan (ref. 1) performed computations for the wall interference problem by the more conventional but far more cumbersome Hankel function series type of kernel formulation. The agreement is excellent in all cases. The method described herein is thus believed to be "checked out" for its validity and feasibility and, thus, to constitute the most reasonable technique for computation of blade-row subsonic oscillatory air forces and moments.

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SYMBOLS

$A_0, A_1, \ldots A_n$	coefficients in pressure-jump expansion
${}_{(0)}A_n, {}_{(1)}A_n$	values of A_n associated with constant and linearly varying upwash, respectively
a	dimensionless elastic-axis location
b	semichord
$C(k)$	Theodorsen flutter function
c	chord
c_0	free-stream sound speed
d^*	slant distance between blades in Prandtl-Glauert space, $\sqrt{s^2 + (s^*)^2/\beta^2}$
$F_m = \beta f_m(\alpha)/\rho_0 b U^2$	
$f_m(\alpha)$	Fourier transform of mth-blade reduced pressure jump
$H_0^{(2)}(\ )$	Hankel function, second kind, order zero
h	translation, downward, of blade elastic axis
$J_n(\ )$	Bessel function, order n
k	reduced frequency, $\omega b/U$
$\overline{L}$	lift amplitude
L_h, L_h'	lift amplitude due to translation; blade in cascade and isolated blade, respectively
L_θ, L_θ'	lift amplitude due to pitch; blade in cascade and isolated blade, respectively
M	free-stream Mach number
$\overline{M}_{ea}$	nose-up moment amplitude about elastic axis

M_h, M_h'	midchord moment amplitude due to translation; in cascade and isolated blade, respectively
$\overline{M}_o$	nose-up moment amplitude about midchord
M_θ, M_θ'	midchord moment amplitude due to pitch; in cascade and isolated blade, respectively
N	magnitude used in replacing infinite integral limits by finite limits
p	pressure
$[p]$	pressure jump, $\left(p_{upper} - p_{lower}\right)$
s	gap distance
s^*	stagger distance
t	time
U	free-stream velocity (relative to blades)
u	x-direction disturbance velocity
v	y-direction disturbance velocity
w_n	coefficients in the expansion of upwash for incompressible case
x,ξ	streamwise coordinates
$x' = x/\beta$	
y,y'	coordinates normal to stream
α,τ	Fourier transform variables
$\beta = \sqrt{1 - M^2}$	
γ	angle utilized in angular transformation of chordwise coordinate
$\eta = \omega/c_o\beta$	
θ	pitch angle
λ	transformation coordinate defined in equation (C1)

$\mu = km/\beta^2$	
ν	integers 0, ±1, ±2, . . .
$\xi' = \xi/\beta$	
ρ_o	free-stream density
σ	interblade phase-lag angle
τ_1^n, τ_2^m	poles of kernel function
ϕ_{L_h}, ϕ_{L_h}'	phase angle between translation and lift due to translation; cascade and isolated blade, respectively
ϕ_{L_θ}	phase angle between pitch and lift due to pitch
ϕ_{M_h}, ϕ_{M_h}'	phase angle between translation and resulting moment in cascade and isolated blade, respectively
ϕ_{M_θ}	phase angle between pitch and moment due to pitch
ψ	reduced pressure function defined in equation (6)
$\Omega = \sigma - \dfrac{\omega M s^*}{c_o \beta^2}$	
ω	frequency
ω_{res}	frequency of lowest resonance condition
$(\bar{\ })$	amplitude, under harmonic-time dependence
$(\ )_1$	real part
$(\ )_2$	imaginary part
$[\ ]$	jump, (Upper - Lower)
sgn	signum function

Subscript:

m	corresponding to mth blade

FORMULATION OF PROBLEM

A development of the linearized aerodynamics problem for the oscillating cascade in subsonic compressible flow is presented as follows. At the outset the system mode shape is introduced whose existence is established in references 3 and 4. This states, in effect, that at flutter the cascade or blade-row system oscillates with but a single characteristic blade mode, this mode occurring with equal amplitude in all blades and with a uniform phase shift between adjacent blades. The ultimate value of this phase shift is, in the flutter of an actual rotor-mounted configuration, that value chosen from among a discrete set of admissible phase-shift values which minimizes flutter velocity. For a linear cascade and, practically, for a rotor-mounted blade row with many blades, it may be assumed that the phase angle is continuously variable and the minimization may be performed with respect to the resulting continuum. It should be noted at this point that the presence of resonance and its dependence upon phase angle in the compressible subsonic case might raise a question with respect to the validity of replacing the minimization problem for a discrete admissible set by a continuum-minimization problem. The fact that resonance will be automatically avoided by the system at flutter, however, indicates that the assumption of continuously variable phase angle, for purposes of flutter-velocity minimization in the many-bladed configuration, should remain valid in the subsonic case. In the event that the assumption of continuously variable phase angle is rendered physically unjustifiable by a trend toward smaller blade numbers, then the minimization process must be performed with respect to the discrete set of admissible phase angles, equal to the number of blades in the blade row.

The blade-row or cascade geometry is shown in figure 1, all variables being physical or dimensional ones. Later scale lengths will be taken with respect to the semichord for convenience. The problem treated is two dimensional, initial blade angles are assumed to be zero or very small, and oscillatory disturbances are considered sufficiently small for linearized theory to apply. Under this theory the equation satisfied by velocity potential and acceleration potential, and hence also by pressure or either of the disturbance velocities, is

$$(1 - M^2)W_{xx} + W_{yy} - \frac{2M}{c_o} W_{xt} - \frac{1}{c_o^2} W_{tt} = 0 \qquad (1)$$

where W may be taken to mean any of the above-mentioned quantities. Writing the equation in terms of pressure and at the same time introducing harmonic time dependence as

$$p(x,y,t) = \bar{p}(x,y)e^{i\omega t} \qquad (2)$$

gives

$$(1 - M^2)\bar{p}_{xx} + \bar{p}_{yy} - 2i\,\frac{M\omega}{c_o}\,\bar{p}_x + \frac{\omega^2}{c_o^2}\,\bar{p} = 0 \tag{3}$$

If the substitution

$$\left.\begin{aligned} x' &= x/\beta \\ y' &= y \end{aligned}\right\} \tag{4}$$

is employed, equation (3) takes the form

$$\bar{p}_{x'x'} + \bar{p}_{y'y'} - \frac{2iM\omega}{c_o\beta}\,\bar{p}_{x'} + \frac{\omega^2}{c_o^2}\,\bar{p} = 0 \tag{5}$$

The function $\psi(x',y')$ is then introduced in accordance with the defining relation

$$\bar{p}(x',y') = \psi(x',y')e^{i\omega Mx'/c_o\beta} \tag{6}$$

In terms of $\psi(x',y')$, equation (5) in turn becomes

$$\psi_{x'x'} + \psi_{y'y'} + \frac{\omega^2}{c_o^2\beta^2}\,\psi = 0 \tag{7}$$

which is simply the familiar reduced wave equation. For convenience the parameter η is introduced, where

$$\eta = \omega/c_o\beta \tag{8}$$

Equation (7) then becomes

$$\psi_{x'x'} + \psi_{y'y'} + \eta^2\psi = 0 \tag{9}$$

At this point it is assumed that frequency ω, hence also η and later reduced frequency $k = \omega b/U$, all have a small negative imaginary part.

$$\left.\begin{aligned} \omega &= \omega_1 + i\omega_2 \\ k &= k_1 + ik_2 \\ \eta &= \eta_1 + i\eta_2 \end{aligned}\right\} \tag{10}$$

where

$$0 < \left\{\begin{matrix} -\omega_2 \\ -k_2 \\ -\eta_2 \end{matrix}\right\} \ll 1$$

The effect of this will be to replace the usual outgoing-wave requirement at infinity by a requirement of boundedness at infinity. The device is a commonly employed one. Ultimately the imaginary part of ω will be forced to zero in a limiting operation.

Singling out for inspection the mth blade, the following notation is introduced:

$$\left.\begin{aligned} y_m' &= y' - ms = y - ms \\ x_m' &= x' - \frac{ms^*}{\beta} = \frac{x - ms^*}{\beta} \end{aligned}\right\} \tag{11}$$

Separation of variables shows that equation (9) has solutions of the form $e^{i\left(\alpha x' \pm y' \sqrt{\eta^2 - \alpha^2}\right)}$ which may be superimposed with respect to the separation parameter α. Then a solution to equation (9) with jump conditions in the proper form for the mth blade is

$$\psi_m(x',y') = \frac{\operatorname{sgn}(y_m')}{2} \int_{\alpha=-\infty}^{\infty} f_m(\alpha) e^{i\left(\alpha x_m' - |y_m'| \sqrt{\eta^2 - \alpha^2}\right)} d\alpha \tag{12}$$

where $f_m(\alpha)$ is an undetermined function. It should be noted that ψ_m as expressed in equation (12) satisfies the differential equation (9) (i.e., differentiation under the integral is permissible) as long as $|y_m'|$ is held away from zero, provided the "proper" branch of the function $\sqrt{\eta^2 - \alpha^2}$ is used and $f_m(\alpha)$ behaves, at worst, algebraically at infinity. This proper branch is the one which behaves like $-i|\alpha|$ as $|\alpha|$ becomes large. Further, the slightly imaginary character of η is needed to keep ψ_m bounded as $|y_m'| \to \infty$ for $|\alpha| < |\eta_1|$.

The jump $[\psi_m]$ in the function ψ_m from the upper to the lower side of the line $y' = ms$ is given by

$$\left[\psi_m(x')\right] = \int_{-\infty}^{\infty} f_m(\alpha) e^{i\alpha x_m'} d\alpha \tag{13}$$

The fact that this jump must vanish for $|x_m'| > \frac{b}{\beta}$ will be taken care of subsequently. Moreover, the functions $f_m(\alpha)$ will be shown ultimately to behave like Bessel functions of argument $\alpha b/\beta$ and hence to approach zero like $1/\sqrt{|\alpha|}$ as $|\alpha| \to \infty$. In view of this and chapter X, section 120, of Carslaw (ref. 5), the integral in equation (13) exists.

If the complete solution ψ to equation (9) is expressed simply as a sum over all blades of functions ψ_m, then

$$\psi = \sum_{m=-\infty}^{\infty} \psi_m \tag{14}$$

Then it is evident that the jump $[\psi]$ in ψ across the line $y = ms$ is given by the term $[\psi_m]$ alone, all other values of ψ_n being continuous across $y = ms$. Now from the phase-shift characteristic of the system mode shape under consideration, it follows that

$$\left[\bar{p}_m(x')\right] = e^{im\sigma}\left[\bar{p}_o\left(x' - \frac{ms^*}{\beta}\right)\right] = e^{im\sigma}\left[\bar{p}_o(x_m')\right] \tag{15}$$

where σ is the angle of phase shift between adjacent blades. Thus, from relation (6) there follows

$$\left[\psi_m(x')\right] e^{i\omega Mx'/c_o\beta} = e^{im\sigma}\left[\psi_o(x_m')\right] e^{i\omega Mx_m'/c_o\beta} \tag{16a}$$

or

$$\left[\psi_m(x')\right] = \left[\psi_o(x_m')\right] e^{im\left(\sigma - \frac{\omega Ms^*}{c_o\beta^2}\right)} \tag{16b}$$

In view of the fact the interblade phase-lag angle σ will appear only in the form in which it enters expression (16b), a new variable Ω is defined as

$$\Omega = \sigma - \frac{\omega Ms^*}{c_o\beta^2} \tag{17}$$

At this point, it is assumed that σ has a small imaginary part σ_2, $\sigma = \sigma_1 + i\sigma_2$, with $\sigma_2 = \omega_2 Ms^*/c_o\beta^2$ in order to keep the quantity Ω real. It should be recalled that ultimately ω_2, hence also σ_2, will be forced to zero. Henceforth, therefore, the quantity Ω is to be considered real. Just as the entire problem is periodic in σ with period 2π, so it will obviously be periodic in Ω with period 2π.

Inserting equations (16) into equation (13) and making use of equation (17) gives

$$\left[\psi_o(x_m')\right] e^{im\Omega} = \int_{-\infty}^{\infty} f_m(\alpha) e^{i\alpha x_m'} \, d\alpha \tag{18}$$

Inverting this expression, considering $\left[\psi_o\right] e^{im\Omega}$ and $f_m(\alpha)$ as Fourier transforms of one another, and noting that $\left[\psi_o(x')\right] = 0$ for $|x'| > \frac{b}{\beta}$ leads to the relation

$$\begin{aligned} f_m(\alpha) &= \frac{e^{im\Omega}}{2\pi} \int_{x_m'=-b/\beta}^{b/\beta} \left[\psi_o(x_m')\right] e^{-i\alpha x_m'} \, dx_m' \\ &= e^{im\Omega} f_o(\alpha) \end{aligned} \tag{19}$$

Thus, the first result of the system-mode-shape concept is that the Fourier transform $f_m(\alpha)$ of the pressure-jump function over the mth blade is equal to that of the pressure-jump function for the zeroth blade multiplied by $e^{im\Omega}$.

Utilizing equation (19) in equation (12) and combining all values of ψ_m, as in equation (14), to form the complete expression for ψ,

$$\psi = \sum_{m=-\infty}^{\infty} e^{im\Omega} \frac{\operatorname{sgn}(y_m')}{2} \int_{-\infty}^{\infty} f_o(\alpha) e^{i\left(\alpha x_m' - |y_m'|\sqrt{\eta^2-\alpha^2}\right)} d\alpha \tag{20}$$

Now, from the second Euler equation of motion, in linearized form, the y-derivative of pressure relates to the material derivative of upwash v as follows:

$$v_t + Uv_x = -\frac{1}{\rho_o} p_y \tag{21}$$

Under harmonic time dependence and in terms of $\left(x' = \frac{x}{\beta}, y' = y\right)$ this becomes

$$i\omega\bar{v} + \frac{U}{\beta} \bar{v}_{x'} = -\frac{1}{\rho_o} \bar{p}_{y'} \tag{22}$$

Equation (22) can be expressed in the form

$$\left(\bar{v}_{x'} + \frac{i\omega}{U}\beta\bar{v}\right) = -\frac{\beta}{\rho_o U}\bar{p}_{y'} \tag{23a}$$

or

$$e^{-i\omega\beta x'/U}\frac{\partial}{\partial x'}\left(\bar{v}e^{i\omega\beta x'/U}\right) = -\beta\bar{p}_{y'}/\rho_o U \tag{23b}$$

In view of the fact that upwash must vanish infinitely far upstream, equations (23) may be integrated to give

$$\bar{v}(x',y') = e^{-i\omega\beta x'/U}\int_{-\infty}^{x'} e^{i\omega\beta\xi'/U}\left(-\frac{\beta}{\rho_o U}\right)\bar{p}_{y'}(\xi',y')\,d\xi' \tag{24}$$

Now expressions (6) and (20) may be utilized in equation (24) to get a relation between the upwash amplitude function $\bar{v}$ and the unknown Fourier transform of the zeroth-blade pressure-jump function as follows:

$$\bar{v}(x',y') = \frac{\beta i e^{-i\omega\beta x'/U}}{2\rho_o U}\int_{\xi'=-\infty}^{x'} e^{i\omega\xi'/U\beta}\sum_{m=-\infty}^{\infty} e^{im\Omega}\int_{\alpha=-\infty}^{\infty} f_o(\alpha)\sqrt{\eta^2 - \alpha^2}\times$$

$$\left[e^{i\left(\alpha\xi_m' - |y_m'|\sqrt{\eta^2-\alpha^2}\right)}\right]d\alpha\,d\xi' \tag{25}$$

where $\xi_m' = \xi' - \frac{ms^*}{\beta}$. Actually, the limiting operation $\lim_{\omega_2\to 0^-}$ or $\lim_{\eta_2\to 0^-}$ or $\lim_{k_2\to 0^-}$ should be understood in equations (24) and (25).

Interchanging the order of integration in equation (25), which is legal as long as $|y_m'| > 0$, the expression for $\bar{v}$ becomes (utilizing the slightly negative imaginary component of ω)

$$\bar{v}(x',y') = \frac{\beta}{2\rho_o U}\lim_{\omega_2\to 0^-} e^{-i\omega\beta x'/U}\int_{\alpha=-\infty}^{\infty} f_o(\alpha)\frac{\sqrt{\eta^2 - \alpha^2}}{\left(\alpha + \frac{\omega}{U\beta}\right)} e^{i\omega x'/U\beta}\times$$

$$\left[\sum_{m=-\infty}^{\infty} e^{im\Omega} e^{i\left(\alpha x_m' - |y_m'|\sqrt{\eta^2-\alpha^2}\right)}\right]d\alpha \tag{26}$$

or

$$\bar{v}(x',y') = \lim_{\omega_2 \to 0^-} \frac{\beta e^{i\omega M^2 x'/U\beta}}{2\rho_o U} \int_{-\infty}^{\infty} \frac{f_o(\alpha)\sqrt{\eta^2 - \alpha^2}}{\left(\alpha + \frac{\omega}{U\beta}\right)} \sum_{m=-\infty}^{\infty} e^{im\Omega} e^{i\left(\alpha x_m' - |y_m'|\sqrt{\eta^2 - \alpha^2}\right)} d\alpha \tag{27}$$

It is next verified that

$$\bar{v}\left(\frac{ns^*}{\beta} + x', ns\right) = e^{in\sigma} \bar{v}(x',0) \tag{28}$$

For, inserting $\left[x' + (ns^*/\beta)\right]$ for x' and ns for y' in equation (27) gives

$$\bar{v}\left(x' + \frac{ns^*}{\beta}, ns\right) = \lim_{\omega_2 \to 0^-} \frac{\beta e^{i\omega M^2 x'/U\beta} e^{i\omega M^2 ns^*/U\beta^2}}{2\rho_o U} \int_{-\infty}^{\infty} \frac{f_o(\alpha)}{\alpha + \frac{\omega}{U\beta}} \sqrt{\eta^2 - \alpha^2}\, e^{i\alpha x'}\, d\alpha \sum_{m=-\infty}^{\infty} e^{im\sigma} e^{-\frac{i\omega M s^*}{c_o \beta^2}} e^{i\left[\alpha\left(\frac{ns^*}{\beta} - \frac{ms^*}{\beta}\right) - |ns - ms|\sqrt{\eta^2 - \alpha^2}\right]} \tag{29}$$

or, letting $\nu = (m - n)$

$$\bar{v}\left(x' + \frac{ns^*}{\beta}, ns\right) = \lim_{\omega_2 \to 0^-} e^{in\sigma} \frac{\beta e^{\frac{i\omega M^2 x'}{U\beta}}}{2\rho_o U} \int_{-\infty}^{\infty} \frac{f_o(\alpha)\sqrt{\eta^2 - \alpha^2}\, e^{i\alpha x'}}{\alpha + \frac{\omega}{U\beta}} \sum_{\nu=-\infty}^{\infty} e^{i\nu\sigma} e^{-\frac{i\omega M s^* \nu}{c_o \beta^2}} e^{i\left(-\frac{\alpha s^* \nu}{\beta} - |\nu| s \sqrt{\eta^2 - \alpha^2}\right)} d\alpha$$

$$= e^{in\sigma} \bar{v}(x',0) \tag{30}$$

is obtained as was to be proven. This result is in accordance with the system-mode-shape concept and reduces the problem, by its periodic nature, from one in which boundary conditions must be inserted on all blades to one in which they need be inserted on only one, say the zeroth blade. Therefore the upwash amplitude distribution $\bar{v}(x')$ on the zeroth blade, which is assumed known for $-\frac{b}{\beta} < x' < \frac{b}{\beta}$, is given by

$$\bar{v}(x',0) = \lim_{\omega_2 \to 0^-} \frac{\beta e^{i\omega M x'/c_o \beta}}{2\rho_o U} \int_{-\infty}^{\infty} \frac{f_o(\alpha)\sqrt{\eta^2 - \alpha^2}}{\alpha + \frac{\omega}{U\beta}} \left[\sum_{m=-\infty}^{\infty} e^{im\left(\Omega - \frac{\alpha s^*}{\beta}\right) - i|m|s\sqrt{\eta^2 - \alpha^2}}\right] e^{i\alpha x'} d\alpha \tag{31}$$

The summation (bracketed expression) may be expressed, in closed form, as

$$\left[\quad\right] = \frac{i \sin\left(s\sqrt{\eta^2 - \alpha^2}\right)}{\cos\left(s\sqrt{\eta^2 - \alpha^2}\right) - \cos\left(\Omega - \frac{\alpha s^*}{\beta}\right)} \tag{32}$$

Utilizing equation (31), equation (30) becomes

$$\bar{v}(x',0) = \bar{v}_0(x') = \lim_{\omega_2 \to 0^-} \frac{i\beta e^{i\omega Mx'/c_0\beta}}{2\rho_0 U} \int_{-\infty}^{\infty} \frac{\sqrt{\eta^2 - \alpha^2}\, f_0(\alpha) e^{i\alpha x'} \sin\left(s\sqrt{\eta^2 - \alpha^2}\right) d\alpha}{\left(\alpha + \frac{\omega}{U\beta}\right)\left[\cos\left(s\sqrt{\eta^2 - \alpha^2}\right) - \cos\left(\Omega - \frac{\alpha s^*}{\beta}\right)\right]} \tag{33}$$

Once again the $1/\sqrt{|\alpha|}$ type behavior of $F_0(\alpha)$ as $|\alpha| \to \infty$ insures the existence of the integral expression for upwash. Now $\bar{v}_0(x')$ is known only for $|x'| < b/\beta$. For $|x'| > b/\beta$ the upwash amplitude function $\bar{v}_0(x')$ is unknown, but the zero-pressure-jump condition is known to hold. Thus, equation (32), considered as an integral equation for $f_0(\alpha)$, constitutes a three-part mixed-boundary-value problem (ref. 6). This exceeds by one the number of parts amenable to Wiener-Hopf technique. Had the blades been doubly infinite in the chordwise direction, then a straightforward use of Fourier transforms would allow solution of equation (33) for $f_0(\alpha)$. For blades semi-infinite in extent, a Wiener-Hopf extension of Fourier transform technique would suffice. The actual problem, with finite blades, exceeds the capabilities of even Wiener-Hopf technique and hence demands some form of variational or numerical treatment.

The approach to be outlined herein will thus constitute a numerical technique for finding what amounts to the Fourier transform of the desired pressure function rather than the pressure function itself. This appears to be a somewhat novel feature of the method. As will be shown subsequently, it will be unnecessary to transform the solution function $f_0(\alpha)$ (or the numerical approximation thereto) in order to get the pressure-distribution function. Rather, it will be shown that oscillatory lift and moment amplitudes are expressible directly in terms of $f_0(\alpha)$ and its first derivative.

The procedure is as follows: From equations (6) and (13), specialized to the zeroth blade,

$$\left[\bar{p}_0(x')\right] = \left[\psi_0(x')\right] e^{i\omega Mx'/c_0\beta} = e^{i\omega Mx'/c_0\beta} \int_{-\infty}^{\infty} f_0(\alpha) e^{i\alpha x'}\, d\alpha \tag{34}$$

At this point, a new dimensionless system of variables is introduced.

$$\left.\begin{aligned} x &= \frac{x'\beta}{b} = \frac{x}{b}\ \text{(Physical)} \\ k &= \omega b/U \\ \tau &= \alpha b/\beta \\ F_o(\tau) &= \frac{\beta f_o(\alpha)}{\rho_o b U^2} = \frac{\beta f_o\left(\frac{\tau\beta}{b}\right)}{\rho_o b U^2} \end{aligned}\right\} \tag{35}$$

Then equation (34) takes the form

$$\frac{\left[\bar{p}_o(x)\right]}{\rho_o U^2} = e^{ikM^2x/\beta^2} \int_{-\infty}^{\infty} F_o(\tau) e^{i\tau x}\, d\tau \tag{36}$$

while equation (33) becomes

$$\frac{\bar{v}_o(x)}{U} = \lim_{k_2 \to 0^-} \frac{i}{2} \beta e^{ikM^2x/\beta^2} \int_{\tau=-\infty}^{\infty} \frac{\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}\, F_o(\tau) e^{i\tau x} \sin \frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}}{\left(\tau + \frac{k}{\beta^2}\right)\left[\cos \frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2} - \cos\left(\Omega - \frac{\tau s^*}{b}\right)\right]}\, d\tau \tag{37}$$

Discussion of the first-order pole at $\tau = -\frac{k}{\beta^2}$ will be delayed until the limiting process $k_1 \to 0^-$ is performed.

It is interesting to note that, despite the presence of the term $\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}$ in equation (37), because of the manner in which the term appears, the entire integrand is single valued or possesses no branch points. This fact will not be used, however, and the original branch choice stated below equation (12) will be followed. This is necessary because the integrand of equation (37) will subsequently be separated into two terms, each of which is itself multivalued or possesses branch points.

LIFT, MOMENT, AND FORM OF TRANSFORM FUNCTION $F_o(\tau)$

Next the statement concerning the direct relation between lift, moment, and the function $F_o(\tau)$ is verified. From equation (36), it follows that

$$F_o(\tau) = \frac{1}{2\pi}\int_{-1}^{1} \frac{\left[\bar{p}_o(x)\right]}{\rho_o U^2} e^{-ikM^2x/\beta^2} e^{-i\tau x} dx \tag{38}$$

Thus, the dimensionless oscillatory lift amplitude is given by

$$\frac{\bar{L}}{\rho_o U^2 b} = -2\pi F_o\left(-\frac{kM^2}{\beta^2}\right) \tag{39}$$

considered positive upward, as in figure 1. Moreover, nose-up moment amplitude about midchord easily follows also from equation (38).

$$\frac{\bar{M}_o}{\rho_o U^2 b^2} = 2\pi i F_o'\left(-\frac{kM^2}{\beta^2}\right) \tag{40}$$

Thus a solution for the Fourier transform $F_o(\tau)$ constitutes, in effect, a complete solution to the problem.

The aerodynamics problem now reduces to the determination of a function $F_o(\tau)$ satisfying equation (37) with $\bar{v}_o(x)$ given in $(-1,1)$ and such that $\left[\bar{p}_o(x)\right]$ vanishes for $|x| > 1$. Referring to equation (36) for $\left[\bar{p}_o(x)\right]$ and to the treatment of Weber's discontinuous integrals in reference 7 it may be seen that an expansion of $F_o(\tau)$ in a series of Bessel functions $J_n(\tau)$ would automatically satisfy the zero-pressure-jump condition off the blade proper (i.e., for $|x| > 1$) Moreover, in view of the expansion formula (ref. 8) for the Bessel functions of a sum-type argument it is evident that an expansion of $F_o(\tau)$ in a series of Bessel functions $J_n(\tau + c)$, where c is a parameter independent of τ, would serve the same purpose. Thus, by assuming $F_o(\tau)$ to be given by an expansion in Bessel functions $J_n(\tau)$ the off-blade-pressure continuity condition is automatically satisfied. All that remains is to determine the coefficients in the expansion such that equation (37) is satisfied to the required degree of approximation on the wing (i.e., for $|x| < 1$).

In order to facilitate the solution to the problem and, at the same time, put $F_o(\tau)$ in a form such that behavior of the integrand in equation (37) may be examined, a trigonometric transformation together with a corresponding pressure expansion, common in both steady and nonsteady two-dimensional wing theory, is introduced as follows:

$$\left.\begin{aligned} x &= -\cos\gamma \\ \frac{\bar{p}_o(x)}{\rho_o U^2} &= A_0 \cot\frac{\gamma}{2} + \sum_1^{\infty} An \sin n\gamma \end{aligned}\right\} \tag{41}$$

Expression (41) for pressure jump contains the proper leading-edge singularity and vanishing trailing-edge value of pressure jump, in accordance with one form of the Kutta condition. Substituting equation (41) into equation (38) leads to an expansion form for $F_o(\tau)$:

$$F_o(\tau) = \frac{1}{2}\left(A_0 + \frac{A_1}{2}\right)J_0\left(\tau + \frac{kM^2}{\beta^2}\right) + i\left(A_0 + \frac{A_2}{2}\right)J_1\left(\tau + \frac{kM^2}{\beta^2}\right) +$$

$$\frac{1}{2}\sum_{\nu=2}^{\infty} (i)^{\nu}\left(A_{\nu+1} - A_{\nu-1}\right)J_{\nu}\left(\tau + \frac{kM^2}{\beta^2}\right) \tag{42}$$

Now, as could be verified by direct integration of equation (41), the dimensionless lift amplitude is given simply by referring to equation (39) and letting $\tau = \frac{-kM^2}{\beta^2}$. Noting that the zeroth-order Bessel function is the only contributor, there results

$$\frac{\bar{L}}{\rho_o U^2 b} = -\pi\left(A_0 + \frac{A_1}{2}\right) \tag{43}$$

Again, referring back to equation (40) and utilizing the Bessel function recurrence properties, the moment is found to be given by the following expression:

$$\frac{\bar{M}_0}{\rho_o U^2 b^2} = \frac{2\pi i}{2}\left[i\left(A_0 + \frac{A_2}{2}\right)\right]J_1{}'(0) = -\frac{\pi}{2}\left(A_0 + \frac{A_2}{2}\right) \tag{44}$$

Once more, this is easily verified by direct weighted integration of equation (41).

The next important point to notice is that, as a result of the assumed expansion equation (41) of the pressure-jump function, $F_o(\tau)$ takes a form, involving Bessel functions, which insures the vanishing of off-blade pressure jump. Thus there remains only to calculate A_0,

A_1, and A_2. Unfortunately, however, as is usually the case in such problems, the higher index coefficients influence these required first three coefficients.

Singularities and Resonance Condition

It should be noted that the integral of equation (37) may possess, in addition to the first-order pole at $\tau = \frac{-k}{\beta^2}$ (which approaches the real axis as $k_2 \to 0^-$), poles at the zeros of the cosine-difference term in the denominator. The form of this term makes it evident that these poles, if they lie on the real axis as $k_2 \to 0^-$, must lie at values of τ so that $|\tau| < \frac{k_1 M}{\beta^2}$. It will now be shown that the arrangement of these poles relates to the criterion for resonance. This is best accomplished by expressing the cosine-difference term in a different form

$$\cos\left(\frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}\right) - \cos\left(\Omega - \frac{\tau s^*}{b}\right) =$$

$$-2 \sin\left(\frac{s\beta}{2b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2} + \frac{\Omega}{2} - \frac{\tau s^*}{2b}\right)\sin\left(\frac{s\beta}{2b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2} - \frac{\Omega}{2} + \frac{\tau s^*}{2b}\right) \tag{45}$$

Letting

$$\left.\begin{aligned} \mu^2 &= k^2M^2/\beta^4 \\ (d^*)^2 &= s^2 + \frac{(s^*)^2}{\beta^2} \end{aligned}\right\} \tag{46}$$

(Note that d^* is the slant distance between blades after a Prandtl-Glauert transformation of the physical plane.) The first sine factor vanishes whenever

$$\frac{s\beta}{2b}\sqrt{\mu^2 - \tau^2} = \pi n - \frac{\Omega}{2} + \frac{\tau s^*}{2b} \tag{47a}$$

for any integer value of n, positive, negative, or zero; the second sine factor vanishes whenever

$$\frac{s\beta}{2b}\sqrt{\mu^2 - \tau^2} = \pi m + \frac{\Omega}{2} - \frac{\tau s^*}{2b} \tag{47b}$$

for integer values of m. Calling the first set of poles $\tau_1{}^n$ and the second set $\tau_2{}^m$, solution of equations (47) gives

$$\tau_1{}^n = \frac{bs^*}{\beta^2(d^*)^2}(\Omega - 2\pi n) + \frac{s}{(d^*)^2}\sqrt{\mu^2(d^*)^2 - \frac{b^2}{\beta^2}(\Omega - 2\pi n)^2} \tag{48a}$$

$$\tau_2{}^m = \frac{bs^*}{\beta^2(d^*)^2}(\Omega + 2\pi m) - \frac{s}{(d^*)^2}\sqrt{\mu^2(d^*)^2 - \frac{b^2}{\beta^2}(\Omega + 2\pi m)^2} \tag{48b}$$

where the poles $\tau_1{}^n$, when complex, lie in the lower half plane and the poles $\tau_2{}^m$, when complex, lie in the upper half plane. It is of interest to note that, as $|n|$ or $|m|$ becomes large,

$$\tau_1{}^n \rightarrow \frac{-2\pi b}{\beta(d^*)^2}\left(\frac{ns^*}{\beta} + i|n|s\right) \tag{49a}$$

$$\tau_2{}^m \rightarrow \frac{2\pi b}{\beta(d^*)^2}\left(\frac{ms^*}{\beta} + i|m|s\right) \tag{49b}$$

In other words, for large values of $|n|$ and $|m|$ the poles approach positions similar to the cascade itself together with its reflection in the y-axis after a Prandtl-Glauert type of transformation. Unfortunately, the possible utilization of residue calculus in the evaluation of the infinite integrals such as appear in equation (37) is defeated by the behavior at infinity of the function $F_o(\tau)e^{i\tau x}$ for $|x| < 1$. Since $F_o(\tau)$ is known to be expressible in an expansion of Bessel functions, it, therefore, acts like $\cos \tau/\sqrt{\tau}$ for large values of τ, and $F_o(\tau)e^{i\tau x}$ fails to converge on either an upper or lower semicircular contour (for $|x| < 1$) as the contour radius approaches infinity. The impossibility of evaluating the infinite integrals via the residue calculus reduces the behavior of the complex poles $\tau_1{}^n$ and $\tau_2{}^m$ to a matter of academic interest only. The numerical solution of the integral equation must be performed with integrations over the real line as they originally occur. Thus the remainder of the analysis will be confined to real-line integrations.

The existence and location of real poles $\tau_1{}^n$ and $\tau_2{}^m$ under certain conditions may be related to the resonance phenomenon in the following manner: For sufficiently small values of $\mu^2(d^*)^2\beta^2/b^2 = k^2M^2(d^*)^2/b^2\beta^2$ there will be a range of values of Ω (recall that only values of Ω in the interval $(0,2\pi)$ need be given consideration because of the periodicity of the entire problem in Ω) in which Ω is bounded away from all multiples of 2π (including zero) far enough to prevent the radicands in equation (48) from becoming positive. For such combinations of Ω, k, M, and d^*/b no real roots are possible. This situation corresponds to the region labelled A in figure 2. As noted in the figure, no real poles of the "cascade term" exist for this zone which will be shown to be below the "lowest resonance point." Now as either of the two boundary lines of zone A are crossed one real root $\tau_1{}^n$ appears, and simultaneously a real root $\tau_2{}^{-n}$ appears. Moreover, they first appear at precisely the same point on the real τ-axis, indicating the presence of a second-order pole. This second-order pole constitutes a nonintegrable singularity and hence suggests a resonance condition. As shown by figure 2, the condition occurs at

$$\Omega = kMd^*/\beta b \tag{50a}$$

or

$$\Omega = 2\pi - (kMd^*/\beta b) \tag{50b}$$

these being the boundaries of zone A. As zones B and C are entered, the real poles persist but separate into two first-order poles. They thus constitute a pair of Cauchy type singularities which are integrable in the sense of Cauchy principal values. Next, the boundaries of zone D are crossed, and two more poles appear on the real axis. Once again, they first appear simultaneously at the same point and form a second-order pole indicating resonance conditions. The boundary between zones C and D is characterized by

$$\Omega = \frac{kMd^*}{\beta b} \tag{51}$$

while that between zones B and D corresponds to

$$\Omega = 2\pi - \frac{kMd^*}{\beta b} \tag{52}$$

Entering into zone D, four poles exist on the real axis, but they are distinct and thus integrable in the Cauchy principal value sense. Proceeding, in this manner, to successively higher values of $\frac{kMd^*}{\beta b}$, more and more new poles appear on the real τ-axis and, as they first appear, they do so in pairs, the pair first occurring at a common point as a zone boundary is crossed. The general equation for the zone boundaries is given by

$$kMd^*/\beta b = \pm(2\pi\nu - \Omega) \tag{53a}$$

or

$$\omega d^*/c_0\beta = \pm(2\pi\nu - \Omega) \tag{53b}$$

with ν an integer, positive, negative, or zero.

Now condition (53) is precisely the set of resonance conditions for a staggered cascade at arbitrary interblade phase angle. For, when the present problem is formulated in terms of the Hankel function series, as given in appendix A, which is then converted via the Poisson summation formula (ref. 9) to an exponential series in much the same way as is performed in reference 1 for the case of tunnel-wall interference, then the resonance conditions appear as relations (53). As is to be expected, these resonance conditions reduce to those of Runyan, Woolston, and Rainey (ref. 2) for the wall-interference problem upon specialization of d^* to s $(s^* = 0)$ and Ω $(\Omega = \sigma$ when $s^* = 0)$ to π. Since resonance, as herein defined, corresponds to a divergent kernel in equation (A1), for all values of x_0, it follows that the pressure jump itself must vanish at the zone boundaries of figure 2. These boundaries are therefore resonance loci for the cascade problem. For the present, the method of computation of aerodynamic oscillatory lift and moment coefficients is confined to zone A, the region lying below the lowest resonance condition.

Computation Methods

The fundamental integral equation (37) is now treated for subresonant conditions (zone A). Referring to equations (31) and (32) it becomes evident that, in the case of a single wing oscillating in subsonic flow, the cascade term

$$\left[\frac{i \sin s\sqrt{\eta^2 - \alpha^2}}{\cos s\sqrt{\eta^2 - \alpha^2} - \cos\left(\Omega - \frac{\alpha s^*}{\beta}\right)}\right] \tag{54}$$

would be replaced simply by 1 (unity). For purposes of computation, the single-wing term is added and subtracted in equation (37) to give

$$\frac{\bar{v}_0(x)}{U} = \text{Single-wing contribution} + \lim_{k_2\to 0^-} \frac{\beta}{2} e^{ikM^2x/\beta^2} \int_{\tau=-\infty}^{\infty} e^{i\tau x}\, d\tau\, \frac{F_0(\tau)\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}}{\tau + \frac{k}{\beta^2}} \left[\frac{i \sin \frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}}{\cos \frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2} - \cos\left(\Omega - \frac{\tau s^*}{b}\right)} - 1\right] \tag{55}$$

The reasons for handling the problem in this way will become evident as the development progresses. The first point to notice is that the bracketed term in equation (55) converges exponentially to zero, thereby effectively reducing the limits of integration from $(-\infty,\infty)$ to quite small finite values for practical computational purposes. In fact if the infinite lower and upper limits are replaced by $-A$ and B, respectively, then the error committed by neglecting the "tails" of the infinite integral may be shown to be bounded by ϵ, where

$$\epsilon = \frac{0.4b}{s\beta} e^{\left[\frac{1}{R} \frac{\beta s}{b}\left(\frac{kM}{\beta^2}\right)^2 - \frac{\beta s R}{b}\right]} \left(1 + \frac{\sqrt{R^2 - \frac{k^2M^2}{\beta^4}}}{R - \frac{k}{\beta^2}}\right) \tag{56}$$

and where R is the smaller of A and B. For reasonable values of M, $\frac{s}{b}$, and k, extremely high accuracy is obtainable with rather small limit magnitude R. For example, with

$$\frac{s^*}{b} = \frac{s}{b} = 2$$

$$M = 0.5$$

$$k = 1$$

it is easily possible to have a subresonant range of Ω, and a value of $R = 8$ gives an error of less than 1×10^{-6}.

Thus, expression (55) is replaced by a numerically equivalent expression in which the finite limits $-A$ and B replace the infinite limits. In particular, it proves convenient to take

$$\left.\begin{aligned} A &= N + \frac{k_1}{\beta^2} \\ B &= N - \frac{k_1}{\beta^2} \end{aligned}\right\} \tag{57}$$

where N is such that the smaller magnitude $N - \frac{k}{\beta^2}$ must satisfy the error requirement (eq. (56)) on R for any prescribed accuracy ϵ. As k/β^2 will seldom in practice exceed 1.5 for $M = 0.5$, it is seen that $N = 9.5$ will be consistent with $R = 8.0$ in the above example. With the limits now centered about the singularity $\tau = \frac{-k}{\beta^2}$, the singularity is easily removed by the artifice of adding and subtracting the value of the remaining portion of the integrand, evaluated at $\tau = \frac{-k}{\beta^2}$. Thus,

$$\frac{\bar{v}_o(x)}{U} = \text{Single-wing contribution} + \lim_{k_2\to 0^-} \frac{\beta}{2} e^{ikM^2x/\beta^2} \int_{-N-\frac{k_1}{\beta^2}}^{N-\frac{k_1}{\beta^2}} \frac{d\tau}{\left(\tau + \frac{k}{\beta^2}\right)} \left\{\left[G(\tau) - G\left(\frac{-k}{\beta^2}\right)\right] + \left[G\left(\frac{-k}{\beta^2}\right)\right]\right\} \tag{58}$$

where

$$G(\tau) = e^{i\tau x} F_o(\tau)\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}\left[\frac{i \sin \frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2}}{\cos \frac{s\beta}{b}\sqrt{\frac{k^2M^2}{\beta^4} - \tau^2} - \cos\left(\Omega - \frac{\tau s^*}{b}\right)} - 1\right] \tag{59}$$

At this point the limiting operation $k_2 \to 0^-$ is introduced. This has no untoward effect on the bracketed term in view of the continuous behavior of the quotient $\left\{\frac{\left[G(\tau) - G\left(\frac{-k}{\beta^2}\right)\right]}{\left(\tau + \frac{k}{\beta^2}\right)}\right\}$ as τ approaches $-k/\beta^2$ (i.e., in this term k is merely replaced by k_1). The expression in brackets $\left[G\left(\frac{-k}{\beta^2}\right)\right]$ in equation (58), however, contributes a term outside the integral as the limiting operation is introduced

$$\lim_{k_2\to 0^-} \int_{-N-\frac{k_1}{\beta^2}}^{N-\frac{k_1}{\beta^2}} \frac{G\left(\frac{-k}{\beta^2}\right)}{\left(\tau + \frac{k}{\beta^2}\right)} d\tau = i\pi G\left(\frac{-k}{\beta^2}\right) + \oint_{-N-\frac{k_1}{\beta^2}}^{N-\frac{k_1}{\beta^2}} \frac{G\left(\frac{-k_1}{\beta^2}\right)}{\left(\tau + \frac{k_1}{\beta^2}\right)} d\tau \tag{60}$$

In equation (60), because of the symmetry of the interval of integration about the point $\tau = \frac{-k_1}{\beta^2}$ and of the antisymmetry of the function $\frac{1}{\tau + \frac{k_1}{\beta^2}}$ in the same range, the integral vanishes and only the external term remains. This explains the reason for symmetrizing the range of integration in equation (57).

Performing these operations and replacing k_1 by k, assumed real henceforth, gives the equation

$$\frac{\bar{v}_o(x)}{U} \doteq \text{Single-wing contribution} + \frac{i\pi}{2}\beta e^{ikM^2x/\beta^2} G\left(\frac{-k}{\beta^2}\right) + \frac{\beta}{2} e^{ikM^2x/\beta^2} \int_{-N-\frac{k}{\beta^2}}^{N-\frac{k}{\beta^2}} \frac{\left[G(\tau) - G\left(\frac{-k}{\beta^2}\right)\right]}{\left(\tau + \frac{k}{\beta^2}\right)} d\tau \qquad (61)$$

Now

$$G\left(\frac{-k}{\beta^2}\right) = e^{ikx/\beta^2} F_o\left(\frac{-k}{\beta^2}\right)\left(\frac{-ik}{\beta}\right)\left[\frac{\sinh \frac{sk}{b}}{\cosh \frac{sk}{b} - \cos\left(\Omega + \frac{ks^*}{b\beta^2}\right)} - 1\right] \qquad (62)$$

Hence, equation (61) becomes

$$\frac{\bar{v}_o(x)}{U} \doteq \text{Single-wing contribution} + \frac{\pi k}{2} F_o\left(\frac{-k}{\beta^2}\right) e^{-ikx}\left[\frac{\sinh \frac{ks}{b}}{\cosh \frac{ks}{b} - \cos\left(\Omega + \frac{ks^*}{\beta^2 b}\right)} - 1\right] + \frac{\beta}{2} e^{ikM^2x/\beta^2} \int_{-N-\frac{k}{\beta^2}}^{N-\frac{k}{\beta^2}} \frac{\left[G(\tau) - G\left(\frac{-k}{\beta^2}\right)\right]}{\left(\tau + \frac{k}{\beta^2}\right)} d\tau \qquad (63)$$

for $|x| < 1$ where it is no longer necessary to indicate the Cauchy principal value since the integrand is now continuous at $\tau = -k/\beta^2$ and where

$$F_o\left(\frac{-k}{\beta^2}\right) = \frac{1}{2}\left[\left(A_0 + \frac{A_1}{2}\right)J_0(-k) + i\left(A_0 + \frac{A_2}{2}\right)J_1(-k) + \frac{1}{2}\sum_{\nu=2}^{\infty} (i)^{\nu}\left(A_{\nu+1} - A_{\nu-1}\right)J_{\nu}(-k)\right] \qquad (64)$$

and where $G(\tau)$ is given by equation (59).

The form of equation (63) is the one recommended for computational purposes. It is to be emphasized that it holds only for the subresonant condition, since when poles $\tau_1{}^n$ and $t_2{}^m$ lie on the real axis as $k_2 \rightarrow 0^-$ (superresonant conditions), then other terms are carried outside the integral in the performance of the limiting operation indicated by equation (60).

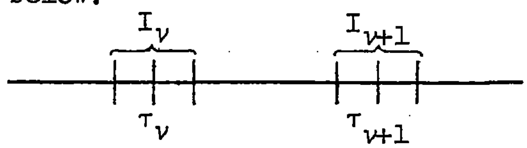
Equation (62) is to be solved for the function $F_o(\tau)$ or, more precisely, for the first three coefficients A_0, A_1, A_2, in the expansion (eq. (42)) of $F_o(\tau)$. Several methods come to mind, but the presence of the single-wing contribution restricts the technique to that in which this contribution is expressible in the required form. In any case, under the assumed expansion (eq. (41)) of the pressure-jump distribution, methods must be used in which the contribution of the single wing to the upwash distribution is expressible in terms of the coefficients A_n of the pressure series. Two means of solving the resulting equation are the pointwise or collocation method and the termwise or Galerkin method. In the former, the desired relation (eq. (62)) is satisfied at a finite number n of discrete points x in the interval $(-1,1)$ and the same number n of coefficients is retained in the pressure series (eq. (41)). The resulting equations are satisfied for the first n coefficients A_m. While only the lowest three values of A are required, nevertheless as more values of A are included in the collocation calculation, the accuracy with which these first three coefficients are calculated increases. This is the method recommended for the subsonic compressible case $(0 < M < 1)$ in view of the existence of numerical data obtained by Woolston and Runyan (ref. 1) for the contribution at each of three points x $(-0.5, 0, \text{and } 0.5)$ to the upwash from the first three terms in a pressure series equivalent to expression (41). In the incompressible case, a Galerkin technique is recommended, in which the error in equation (63) is orthogonalized with respect to a finite number of the complete set of functions 1, $\cos\gamma$, $\cos 2\gamma$, $\cos 3\gamma$, . . . after replacement of x by $(-\cos\gamma)$ throughout equation (63). This procedure is outlined in detail in appendix B.

COMPUTATION METHODS IN SUPERRESONANT CONDITION

Returning to formulation (55) of the aerodynamic integral equation, considerations are extended to the case where zones to the right of the subresonant zone A (fig. 2) are entered. Hence real-axis singularities appear because of the vanishing of the denominator expression (45) in addition to the obvious singularity at $\tau = \frac{-k}{\beta^2}$ in the limit as $k_2 \rightarrow 0^-$. In each of zones B and C, there will be a total of three singularities, and in zone D there will be a total of five, and so on. The computations must, of course, be performed with parameter combinations lying within the interior of a zone, that is, bounded away from the zone boundaries or resonance lines. Hence, in the limit as $k_2 \rightarrow 0^-$, the real poles will be separated by nonvanishing distances. Consider the general case of n real poles τ_ν $(\nu = 1, 2, \ldots n)$ (in the limit as $k_2 \rightarrow 0^-$), one of these being the pole at $\tau = \frac{-k}{\beta^2}$ and the others being the terms $\tau_1{}^n$ and $\tau_2{}^m$ of equations (48). Rewriting equation (55), for convenience, in the form

$$\frac{\bar{v}_0(x)}{U} \doteq \text{Single-wing contribution} + \lim_{k_2 \rightarrow 0^-} \int_{\tau=-N}^{N} H(\tau)\, d\tau \qquad (65)$$

where a value of N is chosen which is sufficiently large to give the required accuracy, centered intervals I_ν are introduced about each pole τ_ν as shown below:



The distinctness of the poles τ_ν insures that nonoverlapping intervals I_ν may be constructed, although this is not crucial to the argument. Outside of the intervals I_ν but within the integration range $(-N,N)$, the cascade contribution (the integral contribution) of equation (64) is calculated by numerical integration in straightforward fashion. Within an interval I_ν centered about the pole τ_ν, the following procedure is followed:

The integrand $H(\tau)$ may be written

$$H(\tau) = H(\tau)(\tau - \tau_\nu)\left(\frac{1}{\tau - \tau_\nu}\right)$$

$$= J(\tau,\tau_\nu)\left(\frac{1}{\tau - \tau_\nu}\right)$$

$$= \frac{J(\tau,\tau_\nu) - J(\tau_\nu,\tau_\nu)}{\tau - \tau_\nu} + \frac{J(\tau_\nu,\tau_\nu)}{\tau - \tau_\nu} \qquad (66)$$

where $J(\tau,\tau_\nu) = H(\tau)(\tau - \tau_\nu)$ is regular at $\tau = \tau_\nu$, since τ_ν is a simple pole of $H(\tau)$. Then, in the limit, as $k_2 \rightarrow 0^-$, the contribution of the term

$$\frac{J(\tau,\tau_\nu) - J(\tau_\nu,\tau_\nu)}{\tau - \tau_\nu} \qquad (67)$$

to the interval I_ν of the integration is simply

$$\int_{I_\nu}\left[\frac{J(\tau,\tau_\nu) - J(\tau_\nu,\tau_\nu)}{\tau - \tau_\nu}\right] d\tau \qquad (68)$$

with real values of τ_ν being understood. As $k_2 \rightarrow 0^-$, the term $\frac{J(\tau_\nu,\tau_\nu)}{\tau - \tau_\nu}$ of equation (65) contributes nothing internally to the integral in view of the symmetry of the interval and the antisymmetry of the function $\frac{1}{\tau - \tau_\nu}$, but a term

$$\pm i\pi J(\tau_\nu, \tau_\nu) \tag{69}$$

appears externally (ref. 10) (outside the integral), the plus sign corresponding to a pole τ_ν which approaches the real axis from above, and the minus sign corresponding to approach from below. The pole at $\tau = \frac{-k}{\beta^2}$ is a special example of the first case, since as $k_2 \rightarrow 0^-$, $\tau = \frac{-k}{\beta^2}$ approaches the real axis from above thereby contributing a term

$$i\pi J\left(\frac{-k}{\beta^2}, \frac{-k}{\beta^2}\right) \tag{70}$$

as was shown in equation (60).

In this fashion, the superresonant cases may be computed by a routine though somewhat more cumbersome procedure.

It is to be noted that for the pole at $\tau = -k/\beta^2$ the operations indicated in equation (66) are actually unnecessary in view of the fact that this singularity is already in the form $\frac{J(\tau, \tau_\nu)}{\tau - \tau_\nu}$. Nevertheless, for the sake of uniformity, it does no harm to include this pole in the general considerations of equation (66).

It will be necessary to know the value of $J(\tau_\nu, \tau_\nu)$ in the limit as $k_2 \rightarrow 0^-$ or as the values of τ_ν become real. Referring to equation (55) for $H(\tau)$ and to expressions (45) through (49) for the values of τ_ν, the following steps may be taken:

$$H(\tau) = \frac{\beta}{2} e^{ix\left(\tau + \frac{kM^2}{\beta^2}\right)} \frac{F_o(\tau)\sqrt{\mu^2 - \tau^2}}{\left(\tau + \frac{k}{\beta^2}\right)} \left[\frac{i \sin\left(\frac{s\beta}{b}\sqrt{\mu^2 - \tau^2}\right)}{-2 \sin\left(\frac{s\beta}{2b}\sqrt{\mu^2 - \tau^2} + \frac{\Omega}{2} - \frac{\tau s^*}{2b}\right)\sin\left(\frac{s\beta}{2b}\sqrt{\mu^2 - \tau^2} - \frac{\Omega}{2} + \frac{\tau s^*}{2b}\right)} - 1\right] \tag{71}$$

Hence,

$$\begin{aligned} J(\tau_\nu, \tau_\nu) &= \lim_{\tau \rightarrow \tau_\nu} H(\tau)(\tau - \tau_\nu) \\ &= \lim_{\tau \rightarrow \tau_\nu} (\tau - \tau_\nu) \frac{\beta}{2} e^{ix\left(\tau + \frac{kM^2}{\beta^2}\right)} \frac{F_o(\tau)\sqrt{\mu^2 - \tau^2}}{\left(\tau + \frac{k}{\beta^2}\right)} \left[\frac{i \sin\left(\frac{s\beta}{b}\sqrt{\mu^2 - \tau^2}\right)}{-2 \sin\left(\frac{s\beta}{2b}\sqrt{\mu^2 - \tau^2} + \frac{\Omega}{2} - \frac{\tau s^*}{2b}\right)\sin\left(\frac{s\beta}{2b}\sqrt{\mu^2 - \tau^2} - \frac{\Omega}{2} + \frac{\tau s^*}{2b}\right)}\right] \end{aligned} \tag{72}$$

In the case where τ_ν is one of the $\tau_1{}^n$ terms of equation (48) with approach to the real axis from below, as $k_2 \rightarrow 0^-$, there results

$$J(\tau_\nu,\tau_\nu) = \frac{\beta}{2} e^{ix\left(\tau_\nu + \frac{kM^2}{\beta^2}\right)} \frac{F_0(\tau_\nu)}{\left(\tau_\nu + \frac{k}{\beta^2}\right)} \left[\frac{bi\sqrt{\mu^2 - \tau_\nu{}^2}}{\left(s^* + \frac{\tau_\nu s\beta}{\sqrt{\mu^2 - \tau_\nu{}^2}}\right)}\right] = \frac{i\beta}{2} be^{ix\left(\tau_\nu + \frac{kM^2}{\beta^2}\right)} \frac{F_0(\tau_\nu)\left(\mu^2 - \tau_\nu{}^2\right)}{\left(\tau_\nu + \frac{k}{\beta^2}\right)} \left(\frac{1}{s^*\sqrt{\mu^2 - \tau_\nu{}^2} + \tau_\nu s\beta}\right) \tag{73}$$

with $\sqrt{\mu^2 - \tau_\nu{}^2} = \frac{b}{s\beta}\left(2\pi n - \Omega + \frac{\tau_\nu s^*}{b}\right)$ and with τ_ν given by equation (48a). Similarly when τ_ν is one of the terms $\tau_2{}^m$ of equation (48b), approaching the real axis from above,

$$J(\tau_\nu,\tau_\nu) = \frac{i\beta}{2} be^{ix\left(\tau_\nu + \frac{kM^2}{\beta^2}\right)} \frac{F_0(\tau_\nu)\left(\mu^2 - \tau_\nu{}^2\right)}{\left(\tau_\nu + \frac{k}{\beta^2}\right)} \left(\frac{1}{\tau_\nu s\beta - s^*\sqrt{\mu^2 - \tau_\nu{}^2}}\right) \tag{74}$$

with $\sqrt{\mu^2 - \tau_\nu{}^2} = \frac{b}{s\beta}\left(2\pi m + \Omega - \frac{\tau_\nu s^*}{b}\right)$ and with τ_ν given by equation (48b).

RESULTS OF SAMPLE CALCULATION FOR ZERO STAGGER, ANTIPHASE CONDITION

In order to establish confidence in the Fourier transform solution technique described herein, a sample set of calculations was carried out completely for a particular condition at which previous results, both experimental and theoretical, exist, the previous theory being that associated with the kernel function in the form of an infinite integral of an infinite series of Hankel functions (appendix A). Runyan, Woolston, and Rainey (ref. 2) present experimental wind-tunnel-wall interference effects on oscillatory lift and moment in pure pitching motion (about midchord) for a series of subsonic Mach numbers with a ratio of tunnel height to wing chord of 3.80. They provide theoretical results, obtained from a three-point collocation treatment of the aerodynamic integral equation with infinite-integral—infinite-series kernel, for these same conditions as well as for pure translatory motion and for other tunnel heights. The tunnel-wall interference problem is, of course, completely equivalent to the blade-row problem under the restriction of zero stagger ($s^* = 0$) and antiphase motion ($\sigma = 180^\circ$).

The present Fourier transform technique, with its simpler, closed-form kernel, was utilized in a three-point collocation solution for the particular case of a blade row at zero stagger under antiphase motion, with a gap-chord ratio of 3.80 and for the particular value of Mach number $M = 0.5$. For this value of Mach number and for the range of values of reduced frequency $(0 < k \leqq 0.7)$ considered in the present example, three-point collocation gives satisfactory accuracy as has been proven by comparison with results obtained from five and seven-point collocation schemes for the isolated wing. The present computation procedure requires the availability of single-wing or isolated-blade contribution to upwash resulting from each term A_n in the pressure-jump expansion as well as the contribution from the remaining blades in the blade row (or the images in the wall-interference interpretation). This single-wing contribution was kindly provided by the Langley Aeronautical Laboratory of the NACA in the form of tables of integrals of the nonsingular portion of the single-wing kernel function with weighting functions corresponding to terms in a pressure-series expansion equivalent to equation (41) herein. The contributions of the singular (logarithmic and first-order pole) terms were calculated and added to the NACA tabular values to provide the data of table I presented herein. This is the tabulation of total single-wing contribution to upwash at three stations (quarter-chord, midchord, and three-quarter-chord) due to each of the first three terms in the pressure-jump expansion (eq. (41)). As noted in the table, the values are for $M = 0.5$ and reduced frequency values k of 0.02, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, and 0.7. In the present case, reference to figure 2 and a brief calculation show that the lowest critical value of k is $k_{crit} = 0.715973$. Table II presents the results of the computation scheme outlined in appendix C for all the remaining blades of the blade row, that is, for the terms exclusive of the single-wing contribution on the right-hand side of equation (63). The corresponding values from tables I and II are simply added to obtain the upwash contribution of each pressure term for the complete blade row or the complete wing-plus-image system in the wall interference interpretation.

Tables III and IV give the results for the single wing and the complete blade row, respectively, of the collocation process where the coefficients ${}_{(0)}A_n$ correspond to a constant unit relative upwash $\left(\frac{\bar{v}_o}{U} \equiv 1\right)$, and the coefficients ${}_{(1)}A_n$ correspond to a linear upwash variation $\left(\frac{\bar{v}_o}{U} = x\right)$ inserted into the left-hand side of equation (63). The expressions below tables III and IV give the dimensionless lift and moment (complex) amplitudes as linear combinations of these A coefficients for arbitrary elastic-axis location and for arbitrary combinations of pitching and translatory motion. The manner in which the systems ${}_{(0)}A_n$ and ${}_{(1)}A_n$ are utilized in computing air forces and moments leads to use of the term "influence coefficients" in describing them.

Figures 3 to 10 illustrate the comparison of lift and moment coefficients derived from the present theory with both the theoretical and experimental results of Runyan, Woolston, and Rainey (ref. 2). These figures relate to either pure pitching motion about midchord or pure translatory motion. It should be emphasized, however, that the data of tables III and IV permit simple calculation of lift and moment for any combination of translation and pitching about any axis. Figure 3 gives the results, in the form of lift amplitude ratio for the blades in cascade or the wing between walls to that of the isolated wing, for pure pitching motion about midchord. The present theory denoted NYU theory is seen to follow closely the theory of reference 2 denoted NACA theory, whereas both theories depart noticeably from the experimental results. The major portion of this discrepancy between theory and experiment probably lies in the pressure-integration technique utilized in reference 2 to obtain lift (and moment) from chordwise pressure-jump measurements. Figure 4 gives theoretical and experimental phase angles for lift in pure pitching about midchord, and once again the NYU and NACA theories follow each other quite closely. The single-wing curve, as shown for purposes of comparison in figure 4, can be obtained from the available single-wing tables or from a simple calculation using table III. Figures 5 and 6 give amplitude ratios and phase angles for moment about midchord in pure pitching about midchord. The two theories give very close agreement, and the experimental amplitudes lie at a nearly constant distance below the theoretical amplitudes, again probably because of the pressure-integration method of obtaining experimental moments. The NACA experimental values and NYU theoretical values of moment phase angle are seen to correspond closely in figure 6 up to just below resonance. As explained in reference 2, the pressure integration technique used therein would be expected to exhibit less severe effects on phase angles than on amplitudes.

Figures 7 to 10 relate to pure translatory wing or blade motion for which NACA theory but no experimental data are available. These figures thus provide simply a comparison of numerical results from the two theories. The agreement is seen in all cases to be excellent, to such a degree, in fact, that the curves are in some regions indistinguishable.

If a generalization were to be attempted regarding the comparison of NYU and NACA theoretical results, it might be to the effect that the NYU results emphasize slightly the "saddleback" shape characteristic of the amplitude-ratio curves and exhibit a sharper decrease in phase angle as the resonant frequency is approached. On the whole, however, the two theories give extremely close results and, thus, each affords an increased measure of confidence in the other.

REMARKS CONCERNING APPLICATION OF METHOD TO BLADE-ROW PROBLEMS

The extremely good agreement between NACA calculations based upon the form of kernel function having an infinite integral and infinite Hankel function series and the NYU results based upon the simple closed-form kernel function stemming from the use of Fourier transform serves to justify the steps taken in the latter theory and to give this method a "clean bill of health" so that it may be used with confidence in the more cumbersome blade-row problems involving nonzero stagger and inter-blade phase-lag angles other than 180°. Comparison of equation (63) of the present report, which includes the general case of arbitrary stagger and arbitrary interblade phase-lag, with expressions in reference 2 for the kernel function in the more restrictive case of zero stagger and antiphase motion serves to illustrate the marked simplicity achieved by the transition to Fourier transform and the subsequent summing of the blade-row series and integration of the semi-infinite integral necessary to convert from pressure to upwash. Furthermore, the exponential decay of the integrand of equation (63) permits setting the finite integration limits to achieve any accuracy desired so long as a quadrature scheme of compatible accuracy is utilized. The use of a three-point collocation technique probably constitutes the weakest link in the accuracy of the present calculations; however, five- or seven-point schemes would require the availability of five- or seven-point tables for the single, isolated wing.

For application to turbine or compressor blade-row flutter or response problems the aerodynamic influence coefficients ${}_{(0)}A_n$ and ${}_{(1)}A_n$ must be computed for an appropriate range of Mach numbers and reduced frequencies and for a range of values of Ω between 0 and 2π (this range corresponds to a shifted range of length 2π for the inter-blade phase angle σ at any particular combination of the parameters ω, M, s^*, and c_o). Critical flutter conditions then correspond to the minimization of flutter velocity with respect to σ as shown in reference 3. Actually, as explained in that reference σ must vary, not continuously in the range 0 to 2π but rather among a discrete set of values depending upon the number of blades in the blade row. However, for large blade numbers, the assumption of continuously varying values of σ is shown in reference 3 to be justified in facilitating the minimization procedure.

The emphasis placed herein on subresonant conditions (fig. 2, zone A) is justified for configurations and operating conditions under which expected values of the parameter $\frac{kMd^*}{\beta b}$ do not exceed a value of unity during the occurrence of flutter. For values of this parameter less than unity, critical conditions will lie in a vertical strip adjacent to the Ω-axis in figure 2, the strip width being less than one-third of the altitude of the zone A triangle. While this in itself does not insure that flutter cannot occur in a superresonant condition (zones B or C), all indications are that it does not. Calculations performed by Lane for the incompressible case $M = 0$ have always indicated a distinct boundedness of critical flutter conditions away from interblade phase angles σ, of 0 and 2π. In the compressible case, the vanishing of self-excitation air forces along the resonance lines aa and bb (fig. 2) precludes the possibility of flutter, in the immediate vicinity of these lines, of a blade row having any internal damping. Hence all indications are that blade-row flutter would occur with values of Ω (note Ω rather than σ in the compressible case) between $(1/4)\pi$ and $(7/4)\pi$ and with critical conditions falling in the subresonant zone A. When values of the parameter in excess of unity are expected, then calculations in the superresonant zones B, C, and D are necessary. The necessity for performing computations in still higher zones (E, F, etc.) appears most unlikely in any case.

CONCLUDING REMARKS

The objective of the work reported herein was to develop a feasible computation technique for the aerodynamic oscillatory lifts and moments experienced by cascades or compressor blade-rows in subsonic flow as well as to examine analytically the existence of the resonance phenomenon and the location or set of conditions at which the kernel function of the aerodynamic integral equation blows up. The objective was achieved by resort to a new technique in which the aerodynamics problem is formulated in terms of the Fourier transform of the blade pressure distributions and numerical solutions are found for the transforms rather than for the pressure distributions proper. It is shown that oscillatory lifts and moments are easily derivable from the transforms without the necessity of inverting. Moreover, the technique permits the expression of the kernel function in a simple closed form, whereas conventional methods lead to an extremely cumbersome kernel consisting of a semi-infinite range integral of a doubly infinite series of Hankel functions of the type encountered in single-wing subsonic oscillatory aerodynamic considerations.

A plot is presented giving the location of resonance conditions, in general, and calculation results are reported for a configuration corresponding to the tunnel-wall interference problem which constitutes a special case of the cascade problem. The conditions for which calculations were performed correspond to the conditions at which the National

Advisory Committee for Aeronautics had previously executed wall interference computations following the conventional integral-equation formulation. The agreement achieved is excellent. The technique developed herein is thus believed to represent a valid and feasible method for obtaining oscillatory cascade or blade-row aerodynamic forces in the subsonic regime.

New York University,
New York, N. Y., August 2, 1956.

APPENDIX A

EXPRESSION OF AERODYNAMIC INTEGRAL EQUATION IN CONVENTIONAL FORM

For the sake of completeness, the integral equation for the pressure-jump amplitude distribution is presented below in a more conventional form, that is, the form involving a kernel expressed as an infinite series of Hankel functions:

$$\frac{\bar{v}_o(x)}{U} = \frac{-i}{4\beta} \sum_{m=-\infty}^{\infty} e^{im\Omega} \int_{x_o=-1}^{1} e^{-ik(x-x_o)} \frac{\left[\bar{p}_o(x_o)\right]}{\rho_o U^2} \int_{\eta=-\infty}^{x-x_o} e^{ik\eta/\beta^2} \frac{\partial^2}{\partial y^2} H_o^{(2)}\left[\frac{kM}{\beta^2}\sqrt{\left(\eta - \frac{ms^*}{b}\right)^2 + \beta^2\left(y - \frac{ms}{b}\right)^2}\right]\Bigg|_{y=0} d\eta\, dx_o \qquad \text{(A1)}$$

where y, η, x, and x_o are all scaled with respect to the semichord b, that is, all are dimensionless. The above integral-equation formulation results from a straightforward application of Green's function for the reduced wave equation to the cascade or blade-row problem under the type of blade-to-blade periodicity predicted by the system mode theory. Once again, the interblade phase-lag angle σ appears only in the form $\left(\sigma - \frac{\omega M s^*}{c_o \beta^2}\right)$ defined as Ω in equation (17).

APPENDIX B

APPLICATION OF AERODYNAMIC INTEGRAL EQUATION

TO INCOMPRESSIBLE FLOW

The aerodynamic integral equation, equation (63), becomes, in the case of incompressible flow ($c_o \to \infty$),

$$\frac{\bar{v}_o(x)}{U} = \text{Single-wing contribution} + \frac{\pi k}{2} F_o(-k) e^{-ikx} \left[\frac{\sinh\left(\frac{ks}{b}\right)}{\cosh \frac{ks}{b} - \cos\left(\sigma + \frac{ks^*}{b}\right)} - 1 \right] + \frac{1}{2} \int_{-N-k}^{N-k} \frac{G_o(\tau) - G_o(-k)}{(\tau + k)} \, d\tau \tag{B1}$$

noting the fact that, as $c_o \to \infty$, $\Omega \to \sigma$. In equation (B1) $G_o(\tau)$ signifies the limit of $G(\tau)$ as $c_o \to \infty$ and is given by

$$G_o(\tau) = -i|\tau| e^{i\tau x} F_o(\tau) \left[\frac{\sinh\left|\frac{s\tau}{b}\right|}{\cosh\left(\frac{s\tau}{b}\right) - \cos\left(\sigma - \frac{\tau s^*}{b}\right)} - 1 \right] \tag{B2}$$

Equation (B1) holds for all values of σ, k, s, and s^* since the possibility of resonance disappears when $c_o \to \infty$. This is evident from examination of figure 2 where, with zero Mach number, the entire problem degenerates to the Ω-axis and only the points $\Omega = 0$ and $\Omega = 2\pi$ (or $\sigma = 0$ and $\sigma = 2\pi$) appear as resonance conditions. Upon closer examination, however, it becomes clear that even these two points do not give resonance. This is because, with vanishing values of Ω or σ, the only possible zero in the cosine difference term (for the incompressible case) is at $\tau = 0$. But at $\tau = 0$ a second-order zero appears in the numerator, completely overpowering the zero in the denominator. It should be noted that, as pointed out by Woolston and Runyan (ref. 1) in the wall-interference problem, resonance is possible in the limit of zero Mach number if this is achieved by letting free-stream velocity go to zero while maintaining finite acoustic velocity. In the present case, however, incompressible flow in the classical sense is implied.

It should be noted that, in the incompressible case,

$$F_o(\tau) = \frac{1}{2}\left[\left(A_0 + \frac{A_1}{2}\right) J_0(\tau) + i\left(A_0 + \frac{A_2}{2}\right) J_1(\tau) + \frac{1}{2} \sum_{\nu=2}^{\infty} \left(A_{\nu+1} - A_{\nu-1}\right) (i)^{\nu} J_{\nu}(\tau)\right] \tag{B3}$$

and $F_o(-k)$ is simply the same expression with $-k$ inserted for τ.

Now referring to Küssner and Schwarz (ref. 11), or to Fung (ref. 12), it is found that in the single-wing case, for zero Mach number, under the trigonometric substitution of equations (41)

$$x = -\cos\gamma$$

$$\frac{p_{upper} - p_{lower}}{\rho_o U^2} = \frac{[p]}{\rho_o U^2} = A_0 \cot\frac{\gamma}{2} + \sum_{n=1}^{\infty} A_n \sin n\gamma$$

If the upwash ratio $\frac{\bar{v}}{U}$ is expressed in a cosine series

$$\frac{\bar{v}(\gamma)}{U} = w_0 + \sum_{n=1}^{\infty} w_n \cos n\gamma \tag{B4}$$

then there is a set of relations giving the pressure coefficients A_n in terms of the (known) upwash coefficients as follows:

$$\left.\begin{aligned}
\frac{A_0}{2} &= C\left(w_0 - \frac{w_1}{2}\right) + \frac{w_1}{2} \\
\frac{A_1}{4} &= \frac{ik}{1}\left(w_0 - \frac{w_2}{2}\right) - \frac{w_1}{2} \\
\frac{A_2}{2} &= \frac{ik}{2}\left(w_1 - w_3\right) - w_2 \\
&\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\frac{A_n}{2} &= \frac{ik}{n}\left(w_{n-1} - w_{n+1}\right) - w_n \qquad (n \geqq 2)
\end{aligned}\right\} \tag{B5}$$

where $C = C(k) = F(k) + iG(k)$ is Theodorsen's well-known flutter function. While equations (B5) constitute an effective solution in the usual sense, for the purposes of this investigation in combining effects with the cascade contribution, it is necessary to back track and find the contribution of each A_n term to all the w_n terms. Eventually coefficients of $\cos n\gamma$ will be equated on both sides of equation (B1) (which is equivalent to the Galerkin type orthogonalization of error with respect to $\cos m\gamma$ where $m = 0, 1, 2, 3, \ldots$) and this requires a knowledge of the contribution of each component of pressure in equations (41) to every upwash component in equation (B4). The method of getting this contribution in the cascade portion is quite straightforward and will be discussed at the conclusion of this appendix. The contribution of each pressure component to the downwash elements in the single-wing case requires a solution of the infinite system of equations (B5) for the terms w_n, assuming the terms A_n to be given. This development follows:

First, it is noticed that the homogeneous system corresponding to equations (B5) is, for $n \geqq 2$,

$$0 = \frac{ik}{n}\left(w_{n-1} - w_{n+1}\right) - w_n \tag{B6}$$

This is satisfied by Bessel functions J_n and Y_n of the first and second kinds, respectively, with argument (-k) and coefficient $(i)^{-n}$ as may be seen by reference to the recurrence relations satisfied by the Bessel functions. Thus a general solution to equation (B6) is

$$w_n = ai^{-n}\,J_n(-k) + bi^{-n}\,Y_n(-k) \tag{B7}$$

Now from equation (B7) a type of Green's function is constructed for the system of equations (B5) by imagining one of the terms A_m to be unity while all others vanish. The corresponding terms w_n are then precisely the desired elements of the single-wing upwash distribution corresponding to the term multiplied by A_m in the pressure expansion. The first two equations of system (B5) do not fall in the pattern of the Bessel function recurrence scheme and thus have the character of initial conditions in a difference equation.

The construction of a Green's function proceeds by assuming first that

$$\left.\begin{aligned} &A_0 = 1 \\ &A_1 = A_2 = A_3 = \ldots 0 \end{aligned}\right\} \tag{B8}$$

in which case

$$\left.\begin{aligned}
\frac{1}{2} &= C\left(w_0 - \frac{w_1}{2}\right) + \frac{w_1}{2} \\
0 &= ik\left(w_0 - \frac{w_2}{2}\right) - \frac{w_1}{2} \\
0 &= \frac{ik}{2}\left(w_1 - w_3\right) - w_2 \\
&\quad \cdots\cdots \\
0 &= \frac{ik}{n}\left(w_{n-1} - w_{n+1}\right) - w_n \qquad (n \geqq 2)
\end{aligned}\right\} \tag{B9}$$

Now let

$$\left.\begin{aligned}
w_1 &= ai^{-1} J_1(-k) + bi^{-1} Y_1(-k) \\
w_2 &= ai^{-2} J_2(-k) + bi^{-2} Y_2(-k) \\
&\quad \cdots\cdots \\
w_n &= ai^{-n} J_n(-k) + bi^{-n} Y_n(-k)
\end{aligned}\right\} \tag{B10}$$

then all except the first two equations of equations (B9) are satisfied, and it remains only to find a, b, and w_0 so that these are satisfied as well. However upon noting that $Y_n(-k)$ blows up as $k \rightarrow 0$, it becomes evident that $b = 0$. Thus only a and w_0 remain to be determined. These follow from the first two equations of system (B9) after substituting from equations (B10) for w_0 and w_1.

$$\left.\begin{aligned}
\frac{1}{2} &= C\left(w_0 + \frac{ai}{2} J_1\right) - \frac{ai}{2} J_1 \\
0 &= ik\left(w_0 + \frac{a}{2} J_2\right) + \frac{ai}{2} J_1
\end{aligned}\right\} \tag{B11}$$

Solving gives

$$w_0^{(0)} = -\frac{a^{(0)}}{2}\left(J_2 + \frac{J_1}{k}\right) \tag{B12a}$$

$$a^{(0)} = \frac{k}{J_1\left[ik(C - 1) - C\right] - CkJ_2} \tag{B12b}$$

the arguments (-k) being understood for the Bessel functions. In equations (B12) the superscript (0) indicates that the values are for the particular case implied by equations (B8). Hence, equations (B10) become, for this case,

$$w_n^{(0)} = a^{(0)}\, i^{-n}\, J_n(-k) \qquad (n \geqq 1) \tag{B13}$$

Note that $w_0^{(0)}$ remains bounded as $k \rightarrow 0$. In fact, $w_0^{(0)} \rightarrow \frac{1}{2C(0)} = \frac{1}{2}$ as $k \rightarrow 0$ which checks the steady-state theory. Next, the term $w_n^{(1)}$ is determined by letting

$$\left.\begin{aligned} A_0 = A_2 = A_3 = A_4 = \ldots 0 \\ A_1 = 1 \end{aligned}\right\} \tag{B14}$$

in the system of equations (B5). All but the first two equations of equations (B10) are satisfied if

$$w_n^{(1)} = a^{(1)}\, J_n(-k) i^{-n} \qquad (n \geqq 1) \tag{B15}$$

again dropping the $Y_n(-k)$ terms for reasons of boundedness of the terms $w_n^{(1)}$ as $k \rightarrow 0$. The two remaining quantities $a^{(1)}$ and $w_0^{(1)}$ must be found so that the first two equations of the system are satisfied. Thus

$$\left.\begin{aligned} 0 &= C\left[w_0^{(1)} + \frac{ia^{(1)}}{2} J_1\right] - \frac{ia^{(1)}}{2} J_1 \\ \frac{1}{4} &= ik\left[w_0^{(1)} + \frac{a^{(1)}}{2} J_2\right] + \frac{ia^{(1)}}{2} J_1 \end{aligned}\right\} \tag{B16}$$

and

$$\left.\begin{aligned}
w_0^{(1)} &= \frac{a^{(1)}}{2}\, iJ_1(-k)\left(\frac{1}{C} - 1\right) \\
a^{(1)} &= \frac{1/2}{J_1(-k)\left(k - \frac{k}{C} + i\right) + ikJ_2(-k)}
\end{aligned}\right\} \tag{B17}$$

which, together with equation (B15) completes this case. Once again, all $w_n^{(1)}$ terms are bounded as $k \to 0$. Continuing in this manner, that is, letting m run through the integers in the relation

$$\left.\begin{aligned}
A_m &= \delta_{mn} = 0 \qquad (m \neq n) \\
A_m &= \delta_{mn} = 1 \qquad (m = n)
\end{aligned}\right\} \tag{B18}$$

it is found for $m = 2$

$$\left.\begin{aligned}
w_n^{(2)} &= a^{(2)}\, i^{-n} J_n(-k) \qquad (n \geqq 2) \\
w_1^{(2)} &= \frac{a^{(2)}\, kJ_2(-k)}{\frac{k}{C} - k - i} \\
w_0^{(2)} &= \frac{w_1^{(2)}}{2}\left(1 - \frac{1}{C}\right) \\
a^{(2)} &= \frac{1}{2J_2(-k) + kJ_3(-k) + \dfrac{ik^2\, J_2(-k)}{\left(\frac{k}{C} - k - i\right)}}
\end{aligned}\right\} \tag{B19}$$

For $m = 3$,

$$\left.\begin{aligned}
w_n^{(3)} &= a^{(3)}\, i^{-n}\, J_n(-k) \qquad (n \geqq 3)\\
w_1^{(3)} &= \frac{ik^2\, a^{(3)}\, J_3(-k)}{k^2 + 2i\left(k - \frac{k}{C} + i\right)}\\
w_0^{(3)} &= \frac{w_1^{(3)}}{2}\left(1 - \frac{1}{C}\right)\\
w_2^{(3)} &= w_1^{(3)}\left(1 - \frac{1}{C} + \frac{i}{k}\right)\\
a^{(3)} &= \left[\frac{3i/2}{3J_3(-k) + kJ_4(-k) - \dfrac{ik^2\left(k - \frac{k}{C} + i\right)J_3(-k)}{k^2 + 2i\left(k - \frac{k}{C} + i\right)}}\right]
\end{aligned}\right\} \qquad \text{(B20)}$$

For $m = 4$,

$$\left.\begin{aligned}
w_n^{(4)} &= a^{(4)}\, i^{-n}\, J_n(-k) \qquad (n \geqq 4)\\
w_1^{(4)} &= \frac{k^3\, a^{(4)}\, J_4(-k)}{k^2\left(k - \frac{k}{C} + i\right) + 3i\left[k^2 + 2i\left(k - \frac{k}{C} + i\right)\right]}\\
w_2^{(4)} &= w_1^{(4)}\left(1 - \frac{1}{C} + \frac{i}{k}\right)\\
w_3^{(4)} &= \frac{w_1^{(4)}}{k^2}\left[k^2 + 2i\left(k - \frac{k}{C} + i\right)\right]\\
w_0^{(4)} &= \frac{w_1^{(4)}}{2}\left(1 - \frac{1}{C}\right)\\
a^{(4)} &= \frac{2}{-kJ_5(-k) - 4J_4(-k) + \dfrac{ik^2\, J_4(-k)\left[k^2 + 2i\left(k - \frac{k}{C} + i\right)\right]}{k^2\left(k - \frac{k}{C} + i\right) + 3i\left[k^2 + 2i\left(k - \frac{k}{C} + i\right)\right]}}
\end{aligned}\right\} \qquad \text{(B21)}$$

Finally, for all values of $m \geqq 2$, a general expression is found for the solution to the system of equations (B5) with $A_n = \delta_{nm}$. Letting

$$\left.\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= \left(k - \frac{k}{C} + i\right) \\
\alpha_2 &= k^2 \alpha_0 + 2i\alpha_1 \\
\alpha_3 &= k^2 \alpha_1 + 3i\alpha_2 \\
&\quad . \; . \; . \; . \; . \; . \; . \; . \\
\alpha_m &= k^2 \alpha_{m-2} + mi\alpha_{m-1}
\end{aligned}\right\} \tag{B22}$$

gives the following general expressions:

$$\left.\begin{aligned}
a^{(m)} &= \left[\frac{\left(\frac{m}{2}\right)(i)^{m-2}}{mJ_m(-k) + kJ_{m+1}(-k) - ik^2 J_m(-k) \frac{\alpha_{m-2}}{\alpha_{m-1}}}\right] \\
w_n^{(m)} &= a^{(m)} (i)^{-n} J_n(-k) \qquad\qquad (n \geqq m) \\
w_1^{(m)} &= \frac{a^{(m)}}{\alpha_{m-1}} k^{m-1} (i)^{-m} J_m(-k) \\
w_0^{(m)} &= \frac{w_1^{(m)}}{2} \left(1 - \frac{1}{C}\right) \\
w_2^{(m)} &= w_1^{(m)} \frac{\alpha_1}{k} \\
w_3^{(m)} &= w_1^{(m)} \frac{\alpha_2}{k^2} \\
w_4^{(m)} &= w_1^{(m)} \frac{\alpha_3}{k^3} \\
&\quad . \; . \; . \; . \; . \; . \; . \; . \\
w_{m-1}^{(m)} &= w_1^{(m)} \frac{\alpha_{m-2}}{k^{m-2}}
\end{aligned}\right\} \tag{B23}$$

Again, all $w_j^{(m)}$ are bounded as $k \to 0$. Thus, the contribution of each single-wing portion of the pressure series (eqs. (41)) to every term in the upwash series (eq. (B4)) has been completely defined. There remains only to express, in correct form, the cascade contribution. This is easily accomplished utilizing the relation (ref. 8)

$$e^{iz \cos \gamma} = J_0(z) - 2\left[J_2(z)\cos 2\gamma - J_4(z)\cos 4\gamma + \ldots\right] +$$

$$2i\left[J_1(z)\cos \gamma - J_3(z)\cos 3\gamma + \ldots\right]$$

$$= \sum_{\nu=0}^{\infty} \epsilon_\nu J_\nu(z)\cos \nu\gamma \tag{B24}$$

where $\epsilon_\nu = 1$ when $\nu = 0$ and $\epsilon_\nu = 2(i)^\nu$ when $\nu > 0$. Applying equation (B24) to equation (B1) and using equations (B3), (B4), and (B12) to (B23) gives

$$\sum_{n=0}^{\infty} w_n \cos n\gamma = \underbrace{\sum_{n=0}^{\infty} \cos n\gamma\left[\sum_{m=0}^{\infty} w_n^{(m)} A_m\right]}_{\text{Single-wing contribution}} +$$

$$\frac{\pi k}{2}\left[\sum_{n=0}^{\infty} \epsilon_n J_n(k)\cos n\gamma\right] F_0(-k)\left[\frac{\sinh \frac{ks}{b}}{\cosh \frac{ks}{b} - \cos\left(\sigma + \frac{ks^*}{b}\right)} - 1\right] +$$

$$\frac{1}{2}\int_{-N-k}^{N-k}\left[\frac{G_0(\tau) - G_0(-k)}{\tau + k}\right] d\tau \tag{B25}$$

with

$$G_0(\tau) = -i|\tau|F_0(\tau)\left[\frac{\sinh\left|\frac{\tau s}{b}\right|}{\cosh \frac{\tau s}{b} - \cos\left(\sigma - \frac{\tau s^*}{b}\right)} - 1\right]\sum_{n=0}^{\infty} \epsilon_n J_n(-\tau)\cos n\gamma \tag{B26a}$$

$$G_0(-k) = -i\left|-k\right|F_o(-k)\left[\frac{\sinh\left|\frac{ks}{b}\right|}{\cosh\frac{ks}{b} - \cos\left(\sigma + \frac{ks^*}{b}\right)} - 1\right]\sum_{n=0}^{\infty}\epsilon_n J_n(k)\cos n\gamma \tag{B26b}$$

The term $F_o(\tau)$ is given by equation (B3) and the term $F_o(-k)$ is given by the statement following equation (B3). The Galerkin process now involves the retention of a finite number p of terms in equations (41) and the equating of coefficients of $\cos n\gamma$ for the first p integers $(n = 0, 1, 2, \ldots (p - 1))$. This constitutes a system of p equations for p unknowns and may be carried as far as desired depending upon the available computing facilities. The major effort involved lies in the computation of the integral

$$\int_{-N-k}^{N-k}\frac{|\tau|}{\tau + k}\left\{J_n(-\tau)J_m(\tau)\left[\frac{\sinh\left|\frac{s\tau}{b}\right|}{\cosh\frac{s\tau}{b} - \cos\left(\sigma - \frac{s^*\tau}{b}\right)} - 1\right] - J_n(k)J_m(-k)\left[\frac{\sinh\left|\frac{sk}{b}\right|}{\cosh\frac{sk}{b} - \cos\left(\sigma + \frac{ks^*}{b}\right)} - 1\right]\right\}d\tau \tag{B27}$$

which, for a sizable number of parameter values, should be performed by a high-speed computer.

APPENDIX C

COMPUTATIONAL PROCEDURES

Following is a description of the computational procedure utilized in evaluating the cascade contribution to the integral equation (63) for purposes of solution by collocation. In order to compare the numerical results of the present method with those of the method of reference 2, the special case of $M = 0.5$, zero stagger, antiphase motion $(\sigma = \Omega = 180^\circ)$, and gap-chord ratio of 3.80 was chosen. Calculations were made for eight values of k which are 0.02, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, and 0.7. All computations were programmed for and executed on the Burroughs E-101 Electronic Digital Computer.

The Burroughs Computer, as currently available at NYU, is of limited capacity having 100 12-digit memory locations and capable of accepting only 112 instructions which are introduced by means of pinboards. To multiply two numbers which are in different memory locations and write the result in a third location requires four instructions. The machine can accomplish approximately four multiplications a second and does not have floating decimal. Hence all problems must be carefully programmed and scaled prior to calculation. It is doubtful that this machine could be used for the general (nonzero-stagger and arbitrary-phase-motion) problem for, as will be shown, these two assumptions introduce several simplifications in the analysis. Even with these simplifications optimum use of the computer capabilities was required.

In order to put the integral of equation (63) in a form more amenable for calculation the following adjustments were made:

(1) To eliminate the necessity of programming square roots, substitutions

$$\left.\begin{aligned} \tau &= \frac{kM}{\beta^2}\cosh\lambda \\ \tau &= \frac{kM}{\beta^2}\sin\lambda \\ \tau &= \frac{-kM}{\beta^2}\cosh\lambda \end{aligned}\right\} \tag{C1}$$

for $\tau > \frac{kM}{\beta^2}$, $\frac{-kM}{\beta^2} \leqq \tau \leqq \frac{kM}{\beta^2}$, and $\tau < \frac{-kM}{\beta^2}$, respectively, were made, and the range of integration $-N - \frac{k}{\beta^2} \leqq \tau \leqq N - \frac{k}{\beta^2}$ was subdivided into the three ranges indicated in equations (C1).

(2) After introducing the factor $\exp\left(\frac{ikM^2x}{\beta^2}\right)$ (see eq. (63)) into the integrand, the integral was separated into real and imaginary parts.

(3) The factor $G\left(-\frac{k}{\beta^2}\right)$ being unnecessary in the two ranges $-\frac{kM}{\beta^2} \leqq \tau \leqq \frac{kM}{\beta^2}$ and $\frac{kM}{\beta^2} \leqq \tau \leqq \left(N - \frac{k}{\beta^2}\right)$, the integral $\int \frac{-G\left(\frac{-k}{\beta^2}\right)}{\tau + \frac{K}{\beta^2}}$ was removed and evaluated analytically for these ranges.

After carrying out these operations and introducing

$$M_1 = \cosh^{-1}\left[\frac{\beta^2}{kM}\left(N - \frac{k}{\beta^2}\right)\right]$$

$$M_2 = \cosh^{-1}\left[\frac{\beta^2}{kM}\left(N + \frac{k}{\beta^2}\right)\right]$$

$$H_1\left(\frac{-k}{\beta^2}\right) = \sin(-kx)F_o\left(\frac{-k}{\beta^2}\right)\frac{\beta}{M}\left[\frac{\sinh\left(\frac{sk}{b}\right)}{\cosh\left(\frac{sk}{b}\right) - \cos\left(\frac{s^*k}{\beta^2 b} + \Omega\right)} - 1\right]$$

$$H_2\left(\frac{-k}{\beta^2}\right) = \cos(kx)F_o\left(\frac{-k}{\beta^2}\right)\frac{\beta}{M}\left[\frac{\sinh\left(\frac{sk}{b}\right)}{\cosh\left(\frac{sk}{b}\right) - \cos\left(\frac{s^*k}{\beta^2 b} + \Omega\right)} - 1\right]$$

the integral in equation (63) was separated into the following parts:

For range $\frac{kM}{\beta^2} \leqq \tau \leqq \left(N - \frac{k}{\beta^2}\right)$,

$$\frac{kM}{2\beta}\int_0^{M_1}\frac{\sinh^2\lambda}{\cosh\lambda + \frac{1}{M}}\sin\left(\frac{kMx}{\beta^2}\cosh\lambda + \frac{kM^2x}{\beta^2}\right)F_o\left(\frac{kM}{\beta^2}\cosh\lambda\right)\left[\frac{\sinh\left(\frac{skM}{b\beta}\sinh\lambda\right)}{\cosh\left(\frac{skM}{b\beta}\sinh\lambda\right) - \cos\left(\frac{s^*kM}{\beta^2 b}\cosh\lambda - \Omega\right)} - 1\right]d\lambda \qquad \text{(C2)}$$

$$\frac{-ikM}{2\beta}\int_0^{M_1}\frac{\sinh^2\lambda}{\cosh\lambda+\frac{1}{M}}\cos\left(\frac{kMx}{\beta^2}\cosh\lambda+\frac{kM^2x}{\beta^2}\right)F_o\left(\frac{kM}{\beta^2}\cosh\lambda\right)\left[\frac{\sinh\left(\frac{skM}{b\beta}\sinh\lambda\right)}{\cosh\left(\frac{skM}{b\beta}\sinh\lambda\right)-\cos\left(\frac{s^*kM}{\beta^2 b}\cosh\lambda-\Omega\right)}-1\right]d\lambda \qquad \text{(C3)}$$

for range $-\frac{kM}{\beta^2}\leqq\tau\leqq\frac{kM}{\beta^2}$,

$$\frac{-kM}{2\beta}\int_{-\pi/2}^{\pi/2}\frac{F_o\left(\frac{kM}{\beta^2}\sin\lambda\right)\cos^2\lambda}{\sin\lambda+\frac{1}{M}}\left[\frac{\sin\left(\frac{kMx}{\beta^2}\sin\lambda+\frac{kM^2x}{\beta^2}\right)\sin\left(\frac{skM}{b\beta}\cos\lambda\right)}{\cos\left(\frac{skM}{b\beta}\cos\lambda\right)-\cos\left(\frac{s^*kM}{\beta^2 b}\sin\lambda-\Omega\right)}+\right.$$

$$\left.\cos\left(\frac{kMx}{\beta^2}\sin\lambda+\frac{kM^2x}{\beta^2}\right)\right]d\lambda \qquad \text{(C4)}$$

$$\frac{ikM}{2\beta}\int_{-\pi/2}^{\pi/2}\frac{F_o\left(\frac{kM}{\beta^2}\sin\lambda\right)\cos^2\lambda}{\sin\lambda+\frac{1}{M}}\left[\frac{\cos\left(\frac{kMx}{\beta^2}\sin\lambda+\frac{kM^2x}{\beta^2}\right)\sin\left(\frac{skM}{b\beta}\cos\lambda\right)}{\cos\left(\frac{skM}{b\beta}\cos\lambda\right)-\cos\left(\frac{s^*kM}{\beta^2 b}\sin\lambda-\Omega\right)}-\right.$$

$$\left.\sin\left(\frac{kMx}{\beta^2}\sin\lambda+\frac{kM^2x}{\beta^2}\right)\right]d\lambda \qquad \text{(C5)}$$

and for range $\left(-N-\frac{k}{\beta^2}\right)\leqq\tau\leqq-\frac{kM}{\beta^2}$,

$$\frac{-kM}{2\beta}\int_0^{M_2}\frac{\sinh\lambda}{\cosh\lambda-\frac{1}{M}}\left\{\sin\left(\frac{-kMx}{\beta^2}\cosh\lambda+\right.\right.$$

$$\left.\frac{kM^2x}{\beta^2}\right)F_o\left(\frac{-kM}{\beta^2}\cosh\lambda\right)\sinh\lambda\left[\frac{\sinh\left(\frac{skM}{b\beta}\sinh\lambda\right)}{\cosh\left(\frac{skM}{b\beta}\sinh\lambda\right)-\cos\left(\frac{s^*kM}{\beta^2 b}\cosh\lambda+\Omega\right)}-1\right]-$$

$$\left.H_1\left(\frac{-k}{\beta^2}\right)\right\}d\lambda \qquad \text{(C6)}$$

$$\frac{ikM}{2\beta} \int_0^{M_2} \frac{\sinh \lambda}{\cosh \lambda - \frac{1}{M}} \left\{ \cos\left(\frac{-kMx}{\beta^2} \cosh \lambda + \right.\right.$$

$$\left.\frac{kM^2x}{\beta^2}\right) F_o\left(\frac{-kM}{\beta^2} \cosh \lambda\right) \sinh \lambda \left[\frac{\sinh\left(\frac{skM}{b\beta} \sinh \lambda\right)}{\cosh\left(\frac{skM}{b\beta} \sinh \lambda\right) - \cos\left(\frac{s^*kM}{\beta^2 b} \cosh \lambda + \Omega\right)} - 1\right] -$$

$$\left. H_2\left(\frac{-k}{\beta^2}\right) \right\} d\lambda \tag{C7}$$

All integrals were evaluated using the Gaussian nine-point integration (ref. 13, p. 285) and were checked by the Newton-Cotes closed-quadrature formula (ref. 13, p. 123). The Gaussian method was accomplished in all cases, except in the nearly resonant case, $k = 0.7$ and $\frac{-\pi}{2} \leqq \lambda \leqq \frac{\pi}{2}$, by subdividing the interval of integration into the nine prescribed points. For the Newton-Cotes method, the interval of integration was divided into 30 equally spaced points and the six-interval formula (ref. 13, p. 123) was used five times. Because of the sharp peaking of the integrand in the above-mentioned case, the interval $\frac{-\pi}{2} \leqq \lambda \leqq \frac{\pi}{2}$ was divided into three segments and the Gaussian nine-point integration was carried out over each segment. This was checked again using the Newton-Cotes six-interval formula over 30 equally spaced points.

In order to evaluate the integrand at each point it was necessary to generate sines, cosines, hyperbolic sines and cosines, and four Bessel functions, J_0, J_1, J_2, and J_3. The standard method of storing coefficients of the power series expansions, as recommended by Burroughs Corporation, would require about 70 of the 100 available memory locations in order that the truncated series used be accurate over the range of arguments of these functions. This was clearly impossible. A method was devised for generating the series by using a recursion formula for the coefficients that would serve for all the above series. For any of these series the nth term is obtainable from the $(n - 1)$ term by multiplication by a factor of the form $\frac{\pm X^2}{2n(2n + H)}$, where X is the argument; the plus sign is used for $\sinh X$ and $\cosh X$ and the minus sign, for all others; $H = 1$ for $\sin X$ and $\sinh X$, -1 for $\cos X$ and $\cosh H$, and 2ν for $J_\nu(X)$.

The case considered of zero stagger ($s^* = 0$) and antiphase motion ($\sigma = \Omega = \pi$) reduces the factor in the integrands of equations (C2), (C3), (C6), and (C7)

$$\frac{\sinh\left(\frac{skM}{b\beta} \sinh \lambda\right)}{\cosh\left(\frac{skM}{b\beta} \sinh \lambda\right) - \cos\left(\frac{s^*kM}{b\beta^2} \cosh \lambda - \Omega\right)} - 1$$

to

$$\frac{\sinh\left(\frac{skM}{b\beta} \sinh \lambda\right)}{\cosh\left(\frac{skM}{b\beta} \sinh \lambda\right) + 1} - 1 \tag{C8}$$

Since the upper limit of integration was fixed by insuring that expression (C8) be sufficiently small, accuracy would be lost by having to subtract the two very close numbers in equation (C8) when λ is near the upper limit. However, this was overcome by using the identity

$$\frac{\sinh X}{\cosh X + 1} - 1 = \frac{-1}{\cosh \frac{X}{2}\left(\sinh \frac{X}{2} + \cosh \frac{X}{2}\right)} \tag{C9}$$

This identity not only permitted more accurate generation of sinh and cosh for smaller arguments $\left(\frac{X}{2} \text{ instead of } X\right)$ but it also retained accuracy in the evaluation of the term (C8).

For the general nonzero stagger and arbitrary phase condition, the problem will clearly not permit this elegant simplification and hence would exceed the capacity of the E-101 computer.

In all calculations (series evaluations and infinite integral truncations) a maximum error of 0.0001 was permitted.

In order to solve the integral equation (63) by collocation at three chordwise stations $\left(x = -\frac{1}{2}, 0, \text{ and } \frac{1}{2}\right)$, these three x-values were treated in the numerical evaluation of the integrals.

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TABLE I

TOTAL SINGLE-WING CONTRIBUTION TO REDUCED UPWASH V_o/U

AT THREE CHORDWISE STATIONS FOR M = 0.5

k	x	A_0 Contribution		A_1 Contribution		A_2 Contribution	
		Real	Imaginary	Real	Imaginary	Real	Imaginary
0.02	0.5	0.4519491	0.0428256	0.22578212	0.02582204	0.21671019	-0.00488926
.02	0	.4515698	.0487572	.00906871	.02734223	.4332256	-.0000756
.02	-.5	.4511555	.0546844	-.20764554	.02597331	.21671191	.00473800
.10	.5	.5352984	.1142825	.26513120	.0809435	.21963380	-.02613512
.10	0	.5304671	.1474325	.04529458	.0901383	.4368039	-.0018872
.10	-.5	.5242158	.1802806	-.17464284	.0847148	.21983553	.02236510
.20	.5	.6450755	.1297442	.31652281	.1172289	.22520583	-.056794409
.20	0	.6338565	.2054147	.09028736	.1397294	.4447680	-.0075312
.20	-.5	.6160410	.2794763	-.13674993	.1322414	.22681011	.041806889
.30	.5	.7510416	.0955288	.36804027	.1334792	.23057684	-.09213552
.30	0	.7379910	.2230042	.1346790	.1734030	.4549473	-.01687482
.30	-.5	.7079702	.3468238	-.10136703	.1669715	.23595639	.058772773
.40	.5	.8455129	.0232180	.41796165	.1350070	.23398967	-.131978574
.40	0	.8394037	.2106121	.178177	.1962812	.4662342	-.0298216
.40	-.5	.7995035	.3925083	-.06791077	.1938359	.24663393	.07355509
.50	.5	.9222588	-.0798821	.46486215	.1245734	.23402381	-.17589716
.50	0	.9352460	.1734976	.2204926	.2107855	.4777981	-.0462385
.50	-.5	.8901497	.42128385	-.03602700	.2150847	.25844802	.08636572
.60	.5	.9761701	-.20748176	.50753505	.1041001	.22949481	-.22325336
.60	0	1.0230507	.1151938	.26134806	.2183600	.4889683	-.0659546
.60	-.5	.9794408	.43592156	-.00549879	.2319856	.27112929	.09736951
.70	.5	1.0031925	-.35347302	.54495248	.0751580	.21942893	-.27323354
.70	0	1.1006352	.0385219	.30047885	.2199862	.4891746	-.0887732
.70	-.5	1.0669196	.43825568	.02381299	.2453494	.28448357	.10669571

TABLE II

EXTERNAL BLADE CONTRIBUTION TO UPWASH V_o/U AT THREE CHORDWISE STATIONS (TOTAL CONTRIBUTION LESS SINGLE-WING CONTRIBUTION)

$$\left[M = 0.5;\ \frac{s}{b} = 7.60;\ s^* = 0;\ \sigma = 180^\circ\right]$$

k (a)	x	A_0 Contribution		A_1 Contribution		A_2 Contribution	
		Real	Imaginary	Real	Imaginary	Real	Imaginary
0.02	0.5	-0.03298	-0.01785	-0.01254	-0.00904	-0.004045	0.000098
.02	0	-.02514	-.01857	-.00847	-.00937	-.004124	.000070
.02	-.5	-.01700	-.01824	-.00440	-.00918	-.004045	.000041
.1	.5	-.07794	-.00621	-.03512	-.00449	-.003992	.001319
.1	0	-.07032	-.00899	-.03108	-.00574	-.004141	.001176
.1	-.5	-.06219	-.01148	-.02695	-.00685	-.004122	.001031
.2	.5	-.09096	.03060	-.04311	.01253	-.002640	.002693
.2	0	-.08608	.02504	-.04018	.00992	-.003055	.002515
.2	-.5	-.08026	.01983	-.03697	.00751	-.003243	.002307
.3	.5	-.08315	.05858	-.04105	.02680	-.000911	.002539
.3	0	-.08215	.05362	-.03982	.02424	-.001556	.002589
.3	-.5	-.07974	.04852	-.03806	.02167	-.002021	.002576
.4	.5	-.07748	.07917	-.03893	.03854	-.000242	.001262
.4	0	-.07796	.07713	-.03829	.03708	-.001056	.001654
.4	-.5	-.07669	.07425	-.03693	.03529	-.001695	.001951
.5	.5	-.08155	.10883	-.04006	.05537	-.001189	-.000562
.5	0	-.08025	.11086	-.03842	.05552	-.002113	.000270
.5	-.5	-.07701	.11119	-.03597	.05489	-.002867	.001021
.6	.5	-.1025	.18419	-.04655	.09567	-.005269	-.002861
.6	0	-.09335	.19156	-.04079	.09772	-.006364	-.001212
.6	-.5	-.08175	.19568	-.03403	.09813	-.007205	.000413
.7	.5	-.24383	.74995	-.08450	.38733	-.03865	-.009206
.7	0	-.16837	.77459	-.04460	.39326	-.04035	-.002671
.7	-.5	-.08955	.78649	-.00403	.39261	-.04099	.003944

[a]Critical k for lowest resonance = 0.715973.

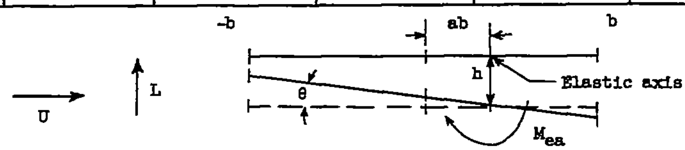
TABLE III

INFLUENCE COEFFICIENTS FOR LIFT AND MOMENT

OF SINGLE WING AT $M = 0.5$

[R indicates real; I indicates imaginary]

k		$_{(0)}A_0$	$_{(0)}A_1$	$_{(0)}A_2$	$_{(1)}A_0$	$_{(1)}A_1$	$_{(1)}A_2$
0.02	R	2.1922	0.00245	0.00021	-0.05923	2.3109	0.00000
.02	I	-.2380	.0604	-.00002	-.1188	-.00055	-.01538
0.1	R	1.7846	0.03141	-0.00397	-.2718	2.3276	0.00037
.1	I	-.5248	.2809	-.00065	-.2551	-.01438	-.07733
0.2	R	1.4902	0.07702	0.01188	-.4285	2.3565	-0.00206
.2	I	-.5737	.5341	-.00309	-.2645	-.04754	-.1559
0.3	R	1.3274	0.1297	0.02527	-0.5218	2.3901	-0.00580
.3	I	-.5688	.7801	-.00744	-.2421	-.09112	-.2358
0.4	R	1.2258	0.1933	0.04357	-0.5848	2.4280	-0.01212
.4	I	-.5610	1.0263	-.01443	-.2148	-.1435	-.3172
0.5	R	1.1555	0.2724	0.06683	-0.6327	2.4703	-0.02148
.5	I	-.5604	1.2751	-.02487	-.1874	-.2054	-.4005
0.6	R	1.1015	0.3716	0.09514	-0.6723	2.5161	-0.03458
.6	I	-.5681	1.5267	-.04001	-.1604	-.2780	-.4857
0.7	R	1.0542	0.4967	0.1347	-0.7081	2.5774	-0.05492
.7	I	-.5833	1.7825	-.0644	-.1278	-.3648	-.6017



$$\bar{L}/(\pi\rho_o bU^2) = \frac{\bar{h}}{b}\left\{ik\left[{}_{(0)}A_0 + \frac{{}_{(0)}A_1}{2}\right]\right\} + \bar{\theta}\left\{(1 - ika)\left[{}_{(0)}A_0 + \frac{{}_{(0)}A_1}{2}\right] + ik\left[{}_{(1)}A_0 + \frac{{}_{(1)}A_1}{2}\right]\right\}$$

$$\bar{M}_{ea}/(\pi\rho_o b^2U^2) = \left[a\bar{L}/(\pi\rho_o bU^2)\right] + \frac{\bar{h}}{b}\left\{\frac{ik}{2}\left[{}_{(0)}A_0 + \frac{{}_{(0)}A_2}{2}\right]\right\} + \bar{\theta}\left\{\left(\frac{1}{2} - \frac{ika}{2}\right)\left[{}_{(0)}A_0 + \frac{{}_{(0)}A_2}{2}\right] + \frac{ik}{2}\left[{}_{(1)}A_0 + \frac{{}_{(1)}A_2}{2}\right]\right\}$$

TABLE IV

INFLUENCE COEFFICIENTS FOR LIFT AND MOMENT OF BLADE

ROW AT ZERO STAGGER, GAPCHORD RATIO = 3.80,

$M = 0.50$, AND $\sigma = \Omega = 180^\circ$

[R indicates real; I indicates imaginary]

k		$_{(0)}A_0$	$_{(0)}A_1$	$_{(0)}A_2$	$_{(1)}A_0$	$_{(1)}A_1$	$_{(1)}A_2$
0.02	R	2.3328	0.08788	0.00217	-0.00957	2.3535	0.00012
.02	I	-.1740	.05666	.00501	-.08549	-.00319	-.01293
0.1	R	2.0249	0.1107	0.00503	-0.1692	2.3674	-0.00028
.1	I	-.6359	.2790	-.0024	-.3090	-.01940	-.07837
0.2	R	1.6234	0.1626	0.01354	-0.3829	2.4008	-0.00245
.2	I	-.7718	.5428	-.00639	-.3630	-.05132	-.1577
0.3	R	1.3932	0.2324	0.02799	-0.5164	2.4472	-0.00629
.3	I	-.7648	.8086	-.01329	-.3410	-.09347	-.2392
0.4	R	1.2637	0.3455	0.04898	-0.6084	2.5167	-0.01322
.4	I	-.7649	1.0831	-.02571	-.3159	-.1526	-.3240
0.5	R	1.1670	0.5459	0.07645	-0.6994	2.6214	-0.02621
.5	I	-.8171	1.3654	-.04978	-.3013	-.2500	-.4148
0.6	R	1.0061	1.0032	0.1043	-0.8538	2.7894	-0.05746
.6	I	-.9864	1.5893	-.1081	-.2883	-.4826	-.5163
0.7	R	-0.1822	1.7376	-0.1246	-1.2242	2.2158	-0.2135
.7	I	-.8314	-.2052	-.2255	.2815	-1.5488	-.5549

$$\bar{L}\Big/\left(\pi\rho_o bU^2\right) = \frac{\bar{h}}{b}\left\{ik\left[_{(0)}A_0 + \frac{_{(0)}A_1}{2}\right]\right\} + \bar{\theta}\left\{(1 - ika)\left[_{(0)}A_0 + \frac{_{(0)}A_1}{2}\right] + ik\left[_{(1)}A_0 + \frac{_{(1)}A_1}{2}\right]\right\}$$

$$\bar{M}_{ea}\Big/\left(\pi\rho_o b^2U^2\right) = \left[a\bar{L}\Big/\left(\pi\rho_o bU^2\right)\right] + \frac{\bar{h}}{b}\left\{\frac{ik}{2}\left[_{(0)}A_0 + \frac{_{(0)}A_2}{2}\right]\right\} + \bar{\theta}\left\{\left(\frac{1}{2} - \frac{ika}{2}\right)\left[_{(0)}A_0 + \frac{_{(0)}A_2}{2}\right] + \frac{ik}{2}\left[_{(1)}A_0 + \frac{_{(1)}A_2}{2}\right]\right\}$$

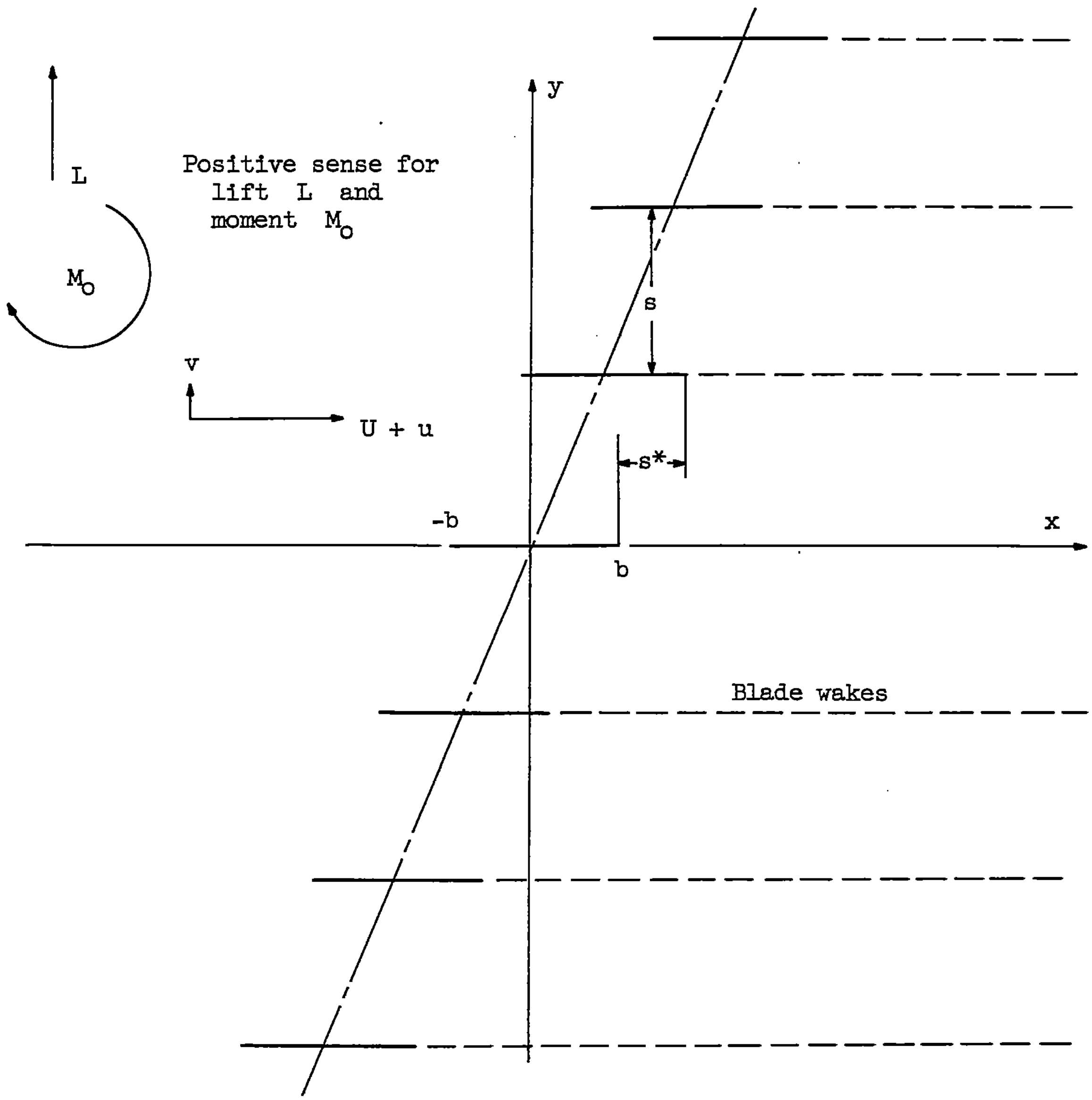


Figure 1.- Cascade geometry.

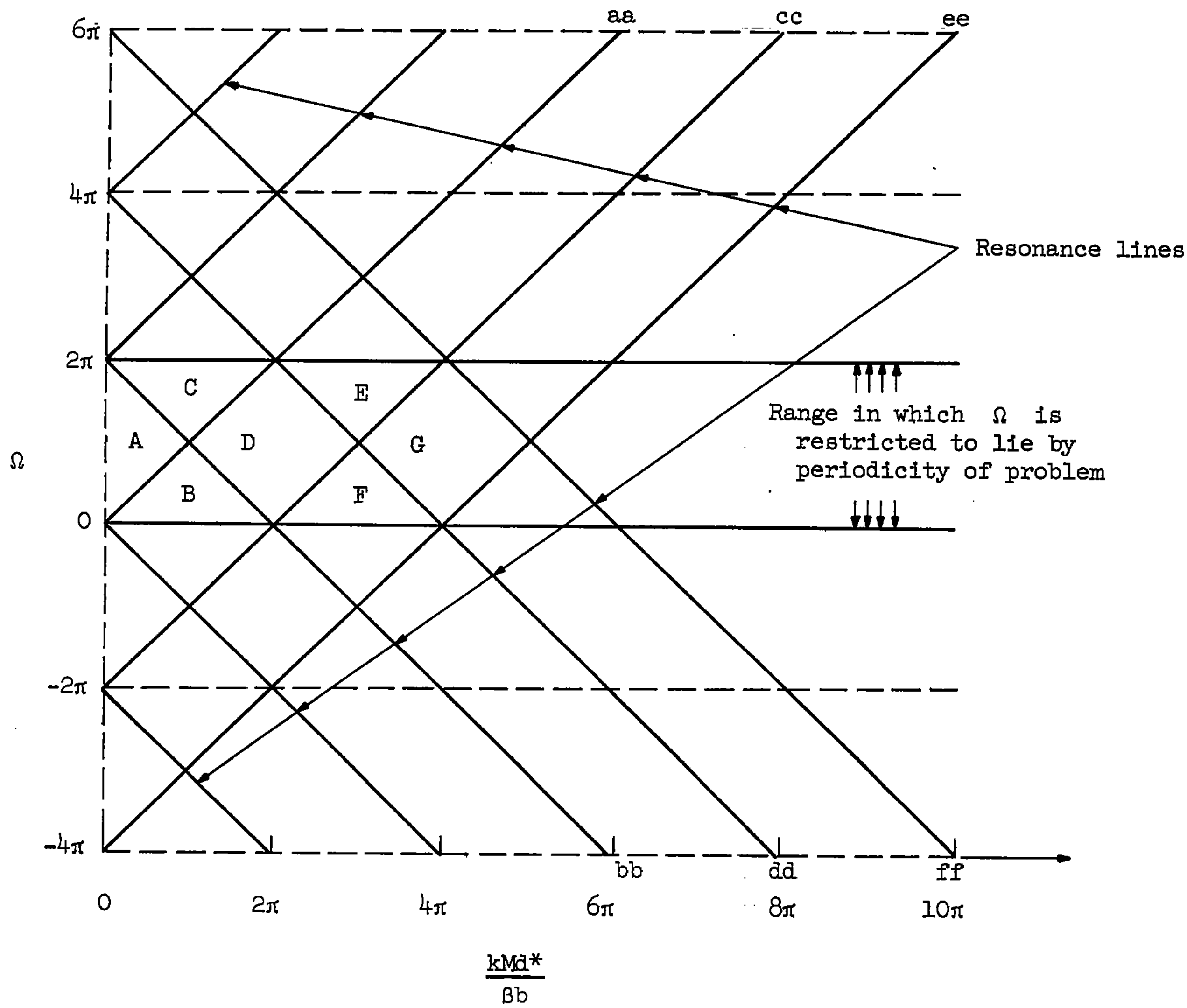


Figure 2.- Relationship between resonance and singularities of cascade term. Letters are defined as follows: zone A, no poles of cascade term lie on real axis in τ-plane; zone B, τ_1^0, τ_2^0 both lie on real axis in τ-plane; zone C, τ_1^1, τ_2^{-1} both lie on real axis; zone D, τ_1^0, τ_2^0, τ_1^1, τ_2^{-1}, lie on real axis; and so forth; line aa, $\tau_1^0 = \tau_2^0$, two poles coincide on real axis; line bb, $\tau_1^1 = \tau_2^{-1}$, two poles coincide on real axis; line cc, $\tau_1^{-1} = \tau_2^1$, two poles coincide on real axis; and so forth.

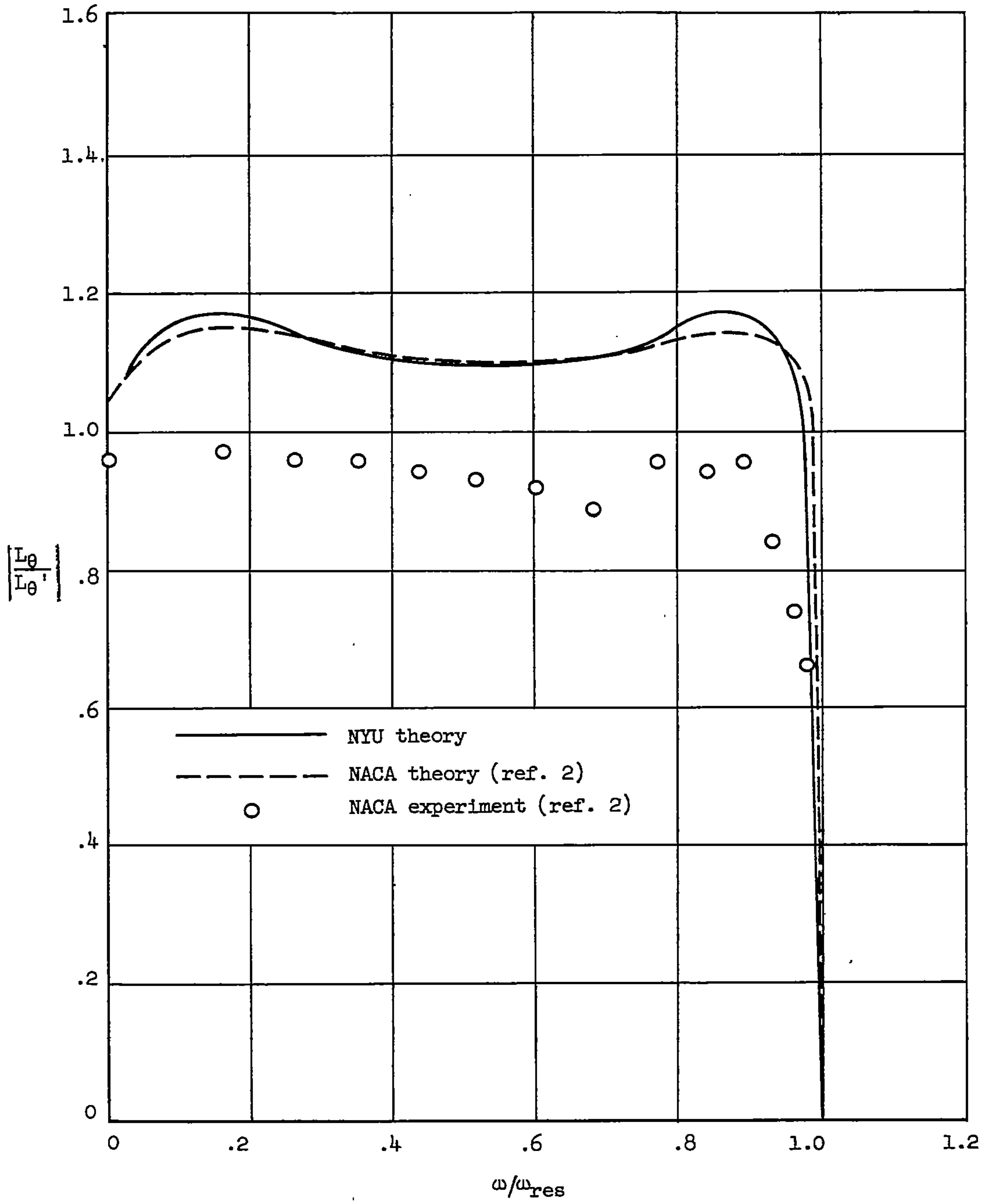


Figure 3.- Effect of cascading or of tunnel walls on lift amplitude for pure pitching motion about midchord. $|L_\theta|$ = Lift amplitude in cascade; $|L_\theta'|$ = Lift amplitude of isolated blade; $\sigma = 180^\circ$; M = 0.5; s/b = 7.60; s* = 0.

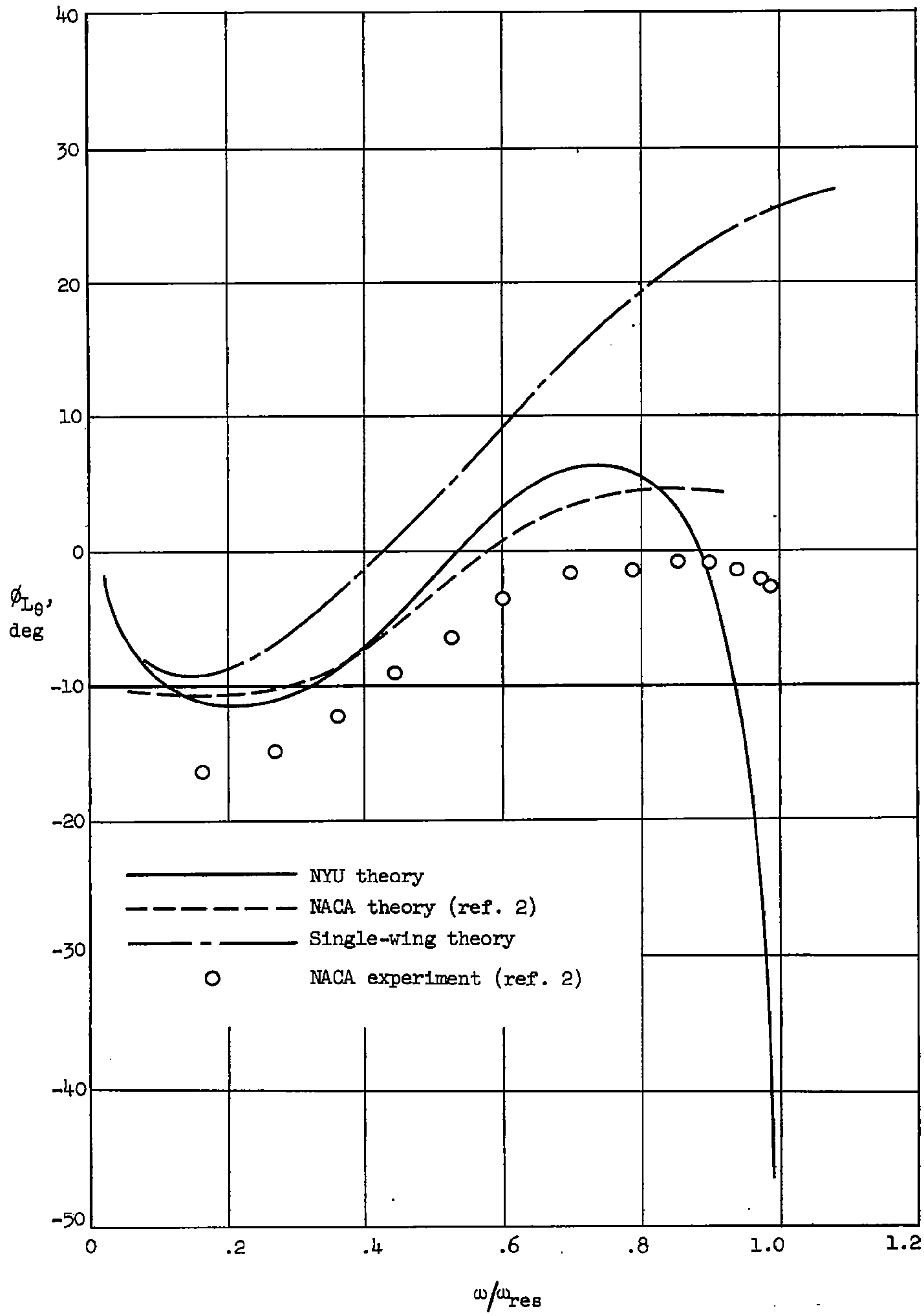


Figure 4.- Effect of cascading or of tunnel walls on lift phase angle for pure pitching motion about midchord. $\sigma = 180^\circ$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.

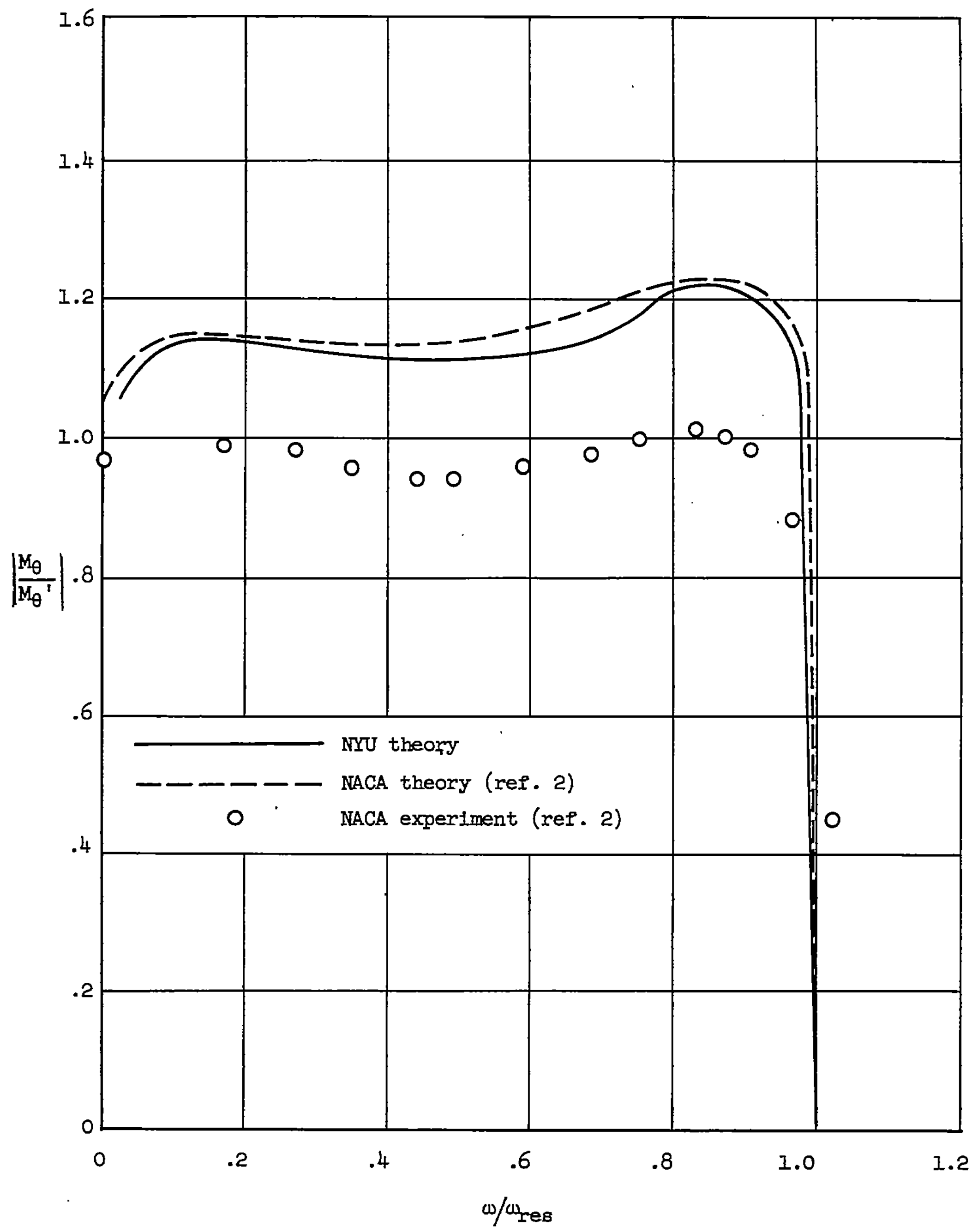


Figure 5.- Effect of cascade or of tunnel walls on moment amplitude for pure pitching about midchord (moment is about midchord). $|M_\theta|$ = Moment amplitude in cascade; $|M_\theta'|$ = Moment amplitude of isolated blade; $\sigma = 180^\circ$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.

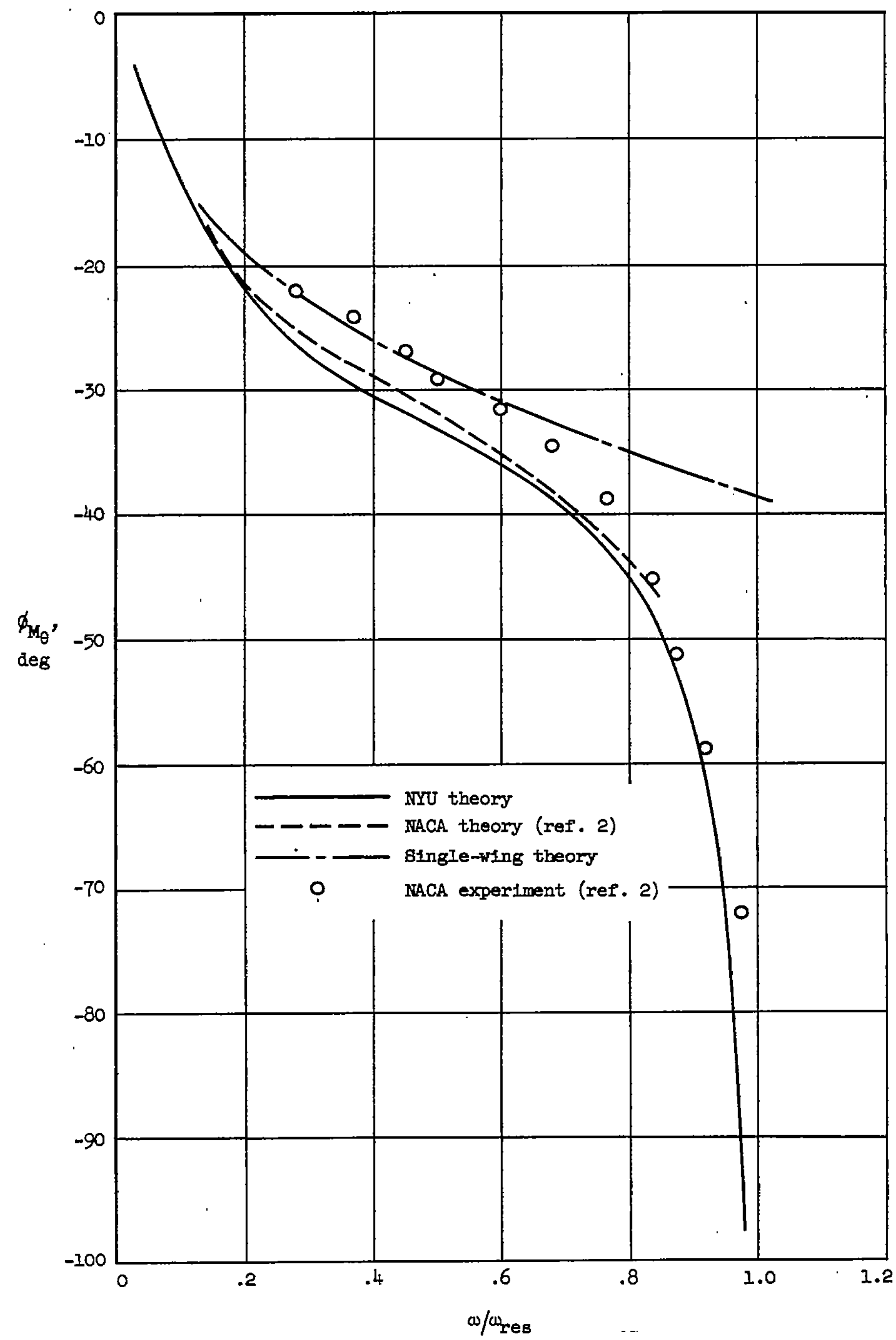


Figure 6.- Effect of cascading or of tunnel walls on moment phase angle for pure pitching about midchord (moment is about midchord). $\sigma = 180^\circ$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.

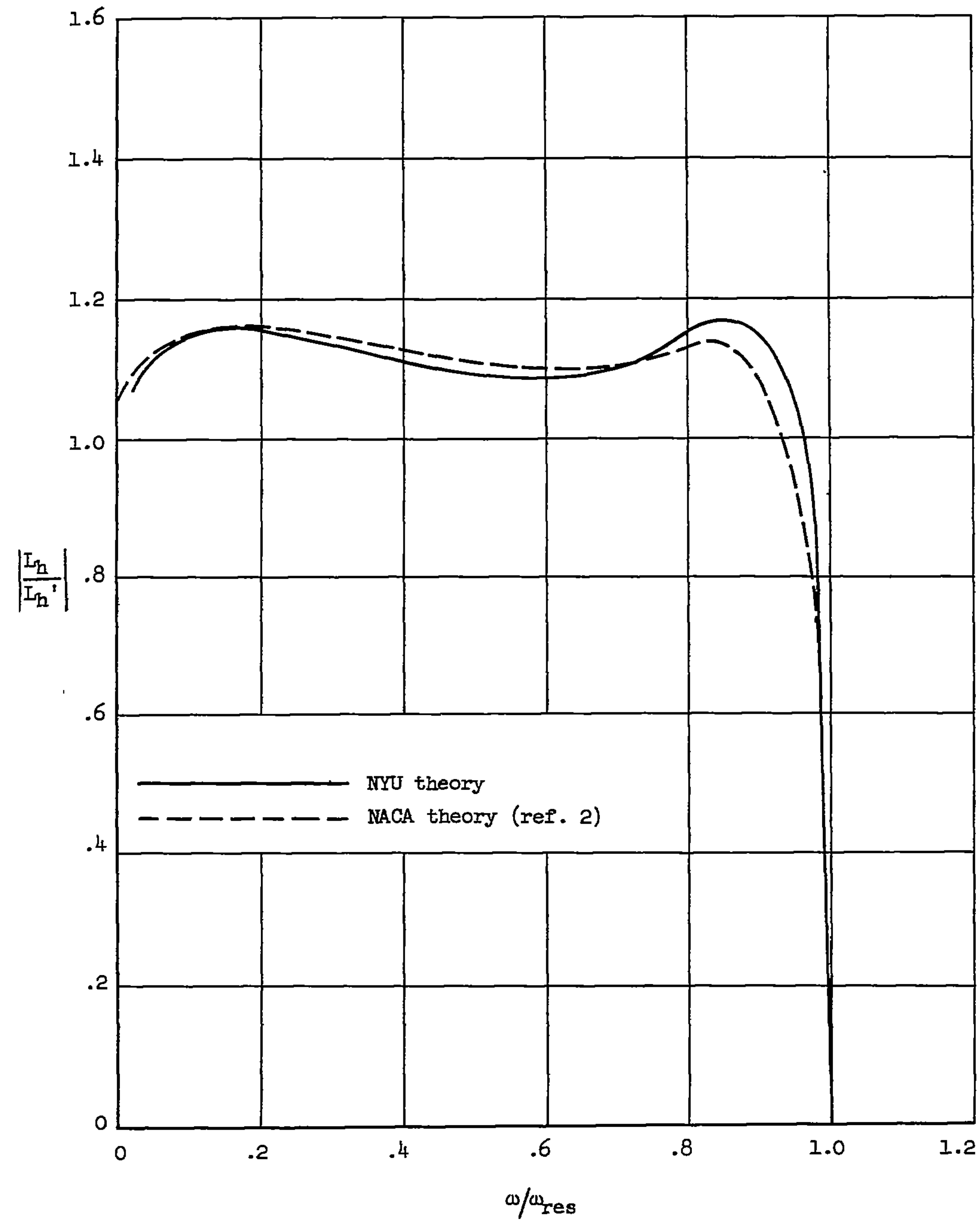


Figure 7.- Effect of cascading or of tunnel walls on lift amplitude for pure translatory motion. $|L_h|$ = Lift amplitude in cascade; $|L_h'|$ = Lift amplitude of isolated blade; $\sigma = 180^\circ$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.

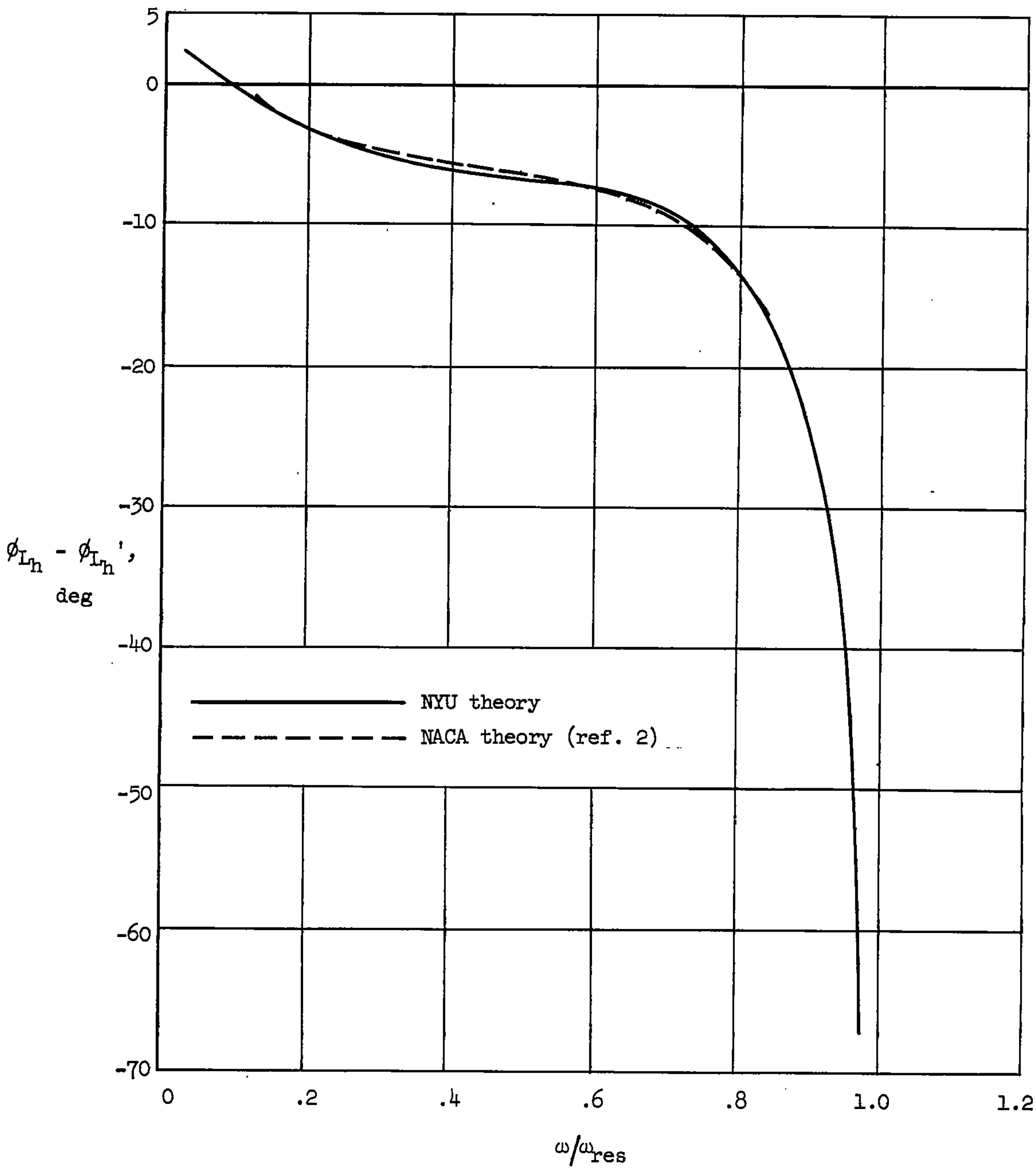


Figure 8.- Effect of cascading or of tunnel walls on lift phase angle for pure translatory motion. ϕ_{L_h} = Lift phase angle in cascade; ϕ_{L_h}' = Lift phase angle of isolated blade; $\sigma = 180^\circ$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.

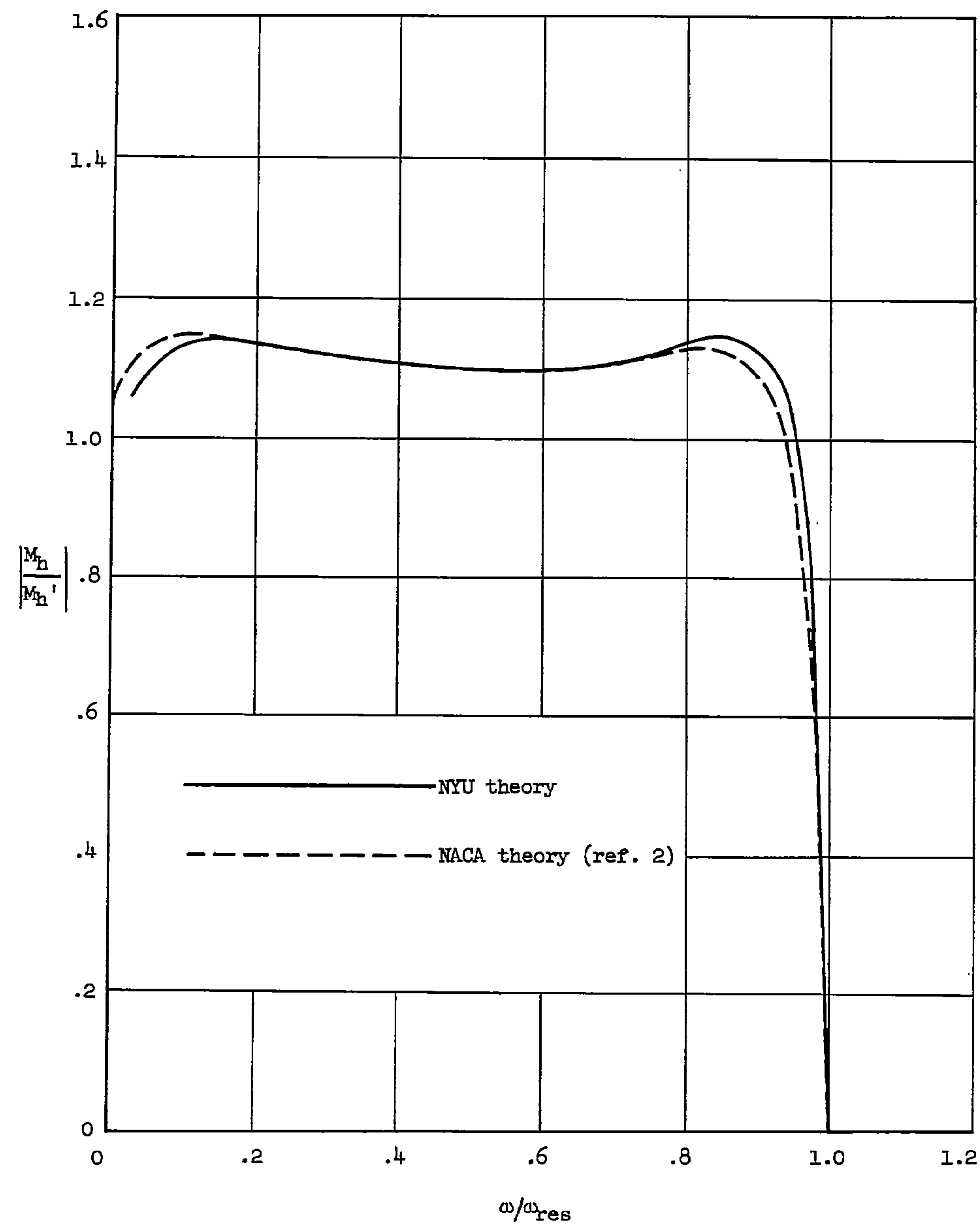


Figure 9.- Effect of cascading or of tunnel walls on moment amplitude for pure translatory motion (moment is about midchord). $|M_h|$ = Cascade moment; $|M_h'|$ = Single-wing moment; $\sigma = 180^\circ$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.

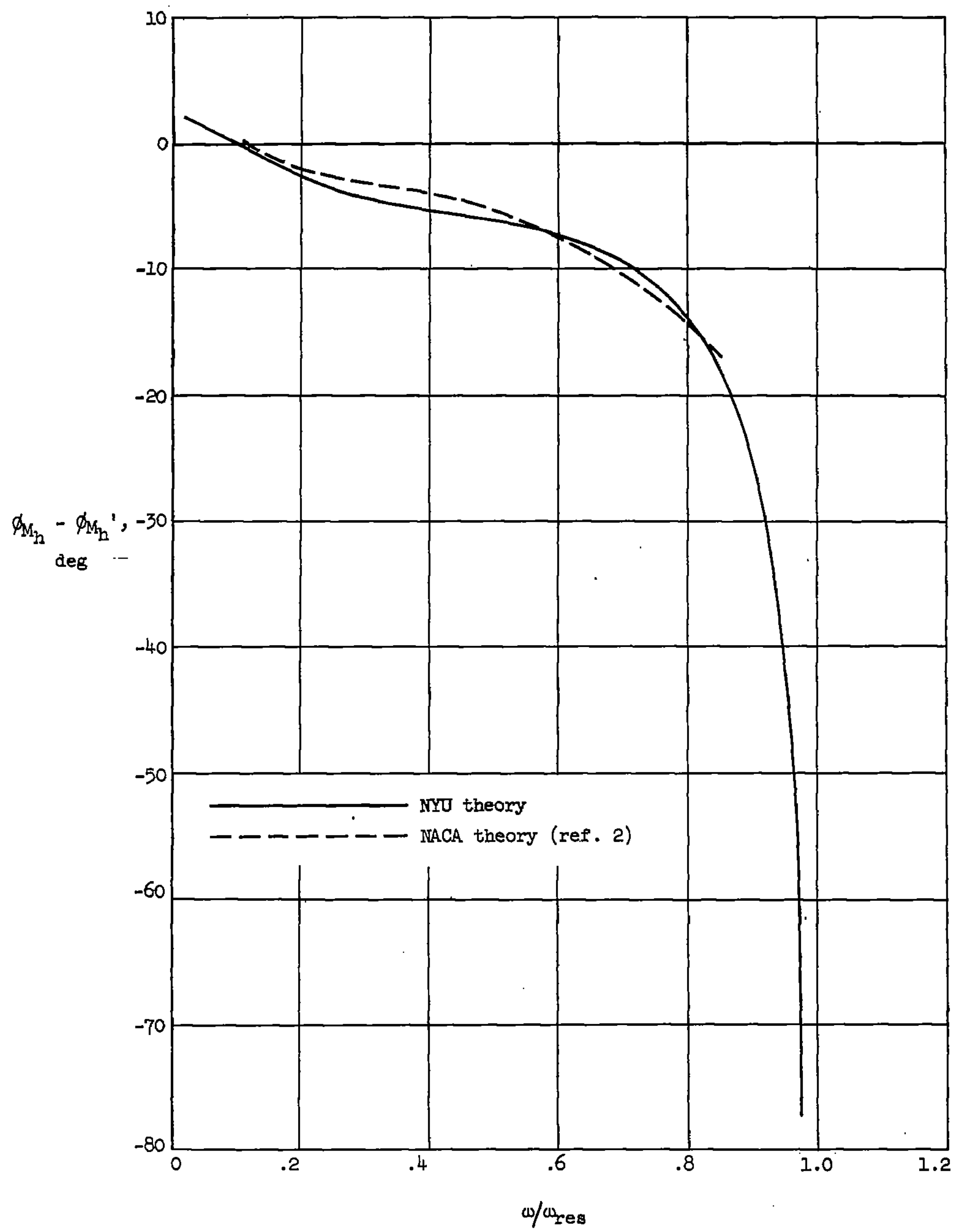


Figure 10.- Effect of cascading or of tunnel walls on moment (about midchord) phase angle for pure translatory motion. ϕ_{M_h} = Cascade moment phase angle; ϕ_{M_h}' = Single-wing moment phase angle; $\sigma = 180^{\circ}$; $M = 0.5$; $s/b = 7.60$; $s^* = 0$.